1 Problems

1. Consider the $\varepsilon-NFA$ defined below:

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>$\emptyset$</td>
<td>${p}$</td>
<td>${q}$</td>
</tr>
<tr>
<td>$q$</td>
<td>${p}$</td>
<td>${q}$</td>
<td>${r}$</td>
</tr>
<tr>
<td>$r$</td>
<td>${q}$</td>
<td>${r}$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

(a) Compute the $\varepsilon$-closure of each state. (3 points)

Solution:

$\varepsilon$-closure$(p) = \{p\}$

$\varepsilon$-closure$(q) = \{p, q\}$

$\varepsilon$-closure$(r) = \{p, q, r\}$

(b) Convert the automaton to a DFA. (4 points)

Solution: □

2. Let $\Sigma = \{a, b, c\}$. Write a regular expression for the language consisting of the set of strings containing at least one $a$ and at least one $b$. (4 points)

Solution: Observe that the simplest approach is to consider those strings in which the first $a$ precedes the first $b$ separately from those where the opposite occurs. The regular expression is:

$c^*a(a + c)^*b(a + b + c)^* + c^*b(b + c)^*a(a + b + c)^*$. □

3. Let $\Sigma = \{0, 1\}$. Which of the following languages is regular? Provide an explanation in each case. (6 points)

(a) $L = \{0^n1^m \mid n \leq m, n, m \geq 0\}$

Proof:
i. Player 1 picks the language $L$ to be proved nonregular, where $L = \{0^n1^m|n \leq m, n, m \geq 0\}$.

ii. Player 2 picks $n$.

iii. Player 1 picks $w = 0^n1^{n+1}$.

iv. Player 2 breaks $w$ into $xyz$, in which $y \neq \epsilon$ and $|xy| \leq n$.

v. Player 1 wins. Since $|xy| \leq n$ and $xy$ comes at the front of $w$, we know that $x$ and $y$ consist of only 0’s. Thus, $y = 0^k$ for $0 < k \leq n$, since $y \neq \epsilon$. The Pumping Lemma tells us that $xy^kz$ is in $L$ if $L$ is regular. If we choose $k = 2$, the resulting string is $w' = 0^{n+2}1^{n+1}$. Clearly $w'$ is not in $L$. Therefore, we have contradicted our assumption that $L$ is regular.

\[\square\]

(b) $L = \{0^n1^m|n \geq m, n, m \geq 0\}$

**Proof:**

i. Player 1 picks the language $L$ to be proved nonregular, where $L = \{0^n1^m|n \geq m, n, m \geq 0\}$.

ii. Player 2 picks $n$.

iii. Player 1 picks $w = 0^n1^n$.

iv. Player 2 breaks $w$ into $xyz$, in which $y \neq \epsilon$ and $|xy| \leq n$.

v. Player 1 wins. We know that $|xy| \leq n$ and $y \neq \epsilon$. Since $xy$ comes at the front of $w$, we know that $x$ and $y$ consist of only 0’s, and that $y$ must contain at least one 0. The Pumping Lemma tells us that $xz$ is in $L$ if $L$ is regular, however, $xz$ has $n$ 1’s, since all of the 1’s of $w$ are in $z$. However, $xz$ also has fewer than $n$ 0’s, because we have lost the 0’s of $y$. Since $y \neq \epsilon$, we know that there can be no more than $n-1$ 0’s among $x$ and $z$. We have assumed $L$ to be a regular language, but have proved that $xz$ is not in $L$. Therefore, we have contradicted our assumption that $L$ is regular.

\[\square\]

(c) $L = \{0^n1^m|n, m \geq 0\}$

**Solution:** Observe that the following regular expression $0^*1^*$ corresponds to $L$. Since we can write a regular expression for $L$, we know that $L$ is regular. \[\square\]

4. Let $\Sigma = \{0, 1\}$. Let $L$ be the language that consists of strings having either 01 repeated one or more times or 010 repeated one or more times. Is $L$ regular? Explain. (4 points)

**Solution:** Observe that $L$ can be written as the following regular expression $((0+1)^*01(0+1)^*01(0+1)^*) + ((0+1)^*010(0+1)^*010(0+1)^*)$. Since we are able to write $L$ as a regular expression, we know that $L$ is regular. (Note each pattern must occur twice in order to be repeated once!) \[\square\]

5. Assume that a regular language $L$ is provided to you as a DFA $A = < Q, \Sigma, \delta, q_0, F >$. How would you check whether $L$ is infinite? (5 points).

*Hint: Pumping Lemma.*

**Proof:** Let $n$ be the Pumping Lemma constant. Test all strings of length between $n$ and $2 \cdot n - 1$ for membership in $L$. If we find even one string, then $L$ is infinite. The reason is that the Pumping Lemma applies to such a string, and it can be “pumped” to show an infinite sequence of strings are in $L$.

Suppose, however, that there are no strings in $L$ whose length is in the range $n$ to $2 \cdot n - 1$. We claim that there are no strings in $L$ of length $2 \cdot n$ or more, and thus there are only a finite number of strings in $L$.

Suppose $w$ is a string in $L$ of length at least $2 \cdot n$, and $w$ is as short as any string in $L$ that has length at least $2 \cdot n$. Then the Pumping Lemma applies to $w$, and we can write $w = xyz$, where $xz$ is also in $L$. How long could $xz$ be? It can’t be as long as $2 \cdot n$, because it is shorter than $w$, and $w$ is as short as any string in $L$ of length $2 \cdot n$ or more. Secondly, $|z| \geq n$ and hence $|xz| \geq n$. Thus, $xz$ is of length between $n$ and $2 \cdot n - 1$, which is a contradiction, since we assumed that there were no strings in $L$ with a length in that range. \[\square\]

6. Let $\Sigma = \{0, 1\}$. We showed in class that the language $L = \{0^n1^n|n \geq 0\}$ is not regular. Argue using closure properties of regularity, that $L' = \{0^i1^j|i \neq j\}$ is not regular. (4 points)
**Proof:** Start out by complementing the language \( L' \); the resulting language is the language consisting of all strings of 0’s and 1’s that are not in \( 0^*1^* \), plus the strings in \( L \). Now, if we intersect the complement of \( L' \) with \( 0^*1^* \), the result is precisely the language \( L \). Since complementation and intersection with a regular set preserve regularity, if the given language were regular, then so would be \( L \). We already know that \( L \) is not regular, therefore, we can conclude that the given language \( L' \) is not regular. \( \square \)