An Introduction to First-Order Logic

K. Subramani

\(^1\)Lane Department of Computer Science and Electrical Engineering
West Virginia University

Completeness, Compactness and Inexpressibility
1. Completeness of proof system for First-Order Logic
   - The notion of Completeness
   - The Completeness Proof

2. Consequences of the Completeness theorem
   - Complexity of Validity
   - Compactness
   - Model Cardinality
   - Löwenheim-Skolem Theorem
   - Inexpressibility of Reachability
1 Completeness of proof system for First-Order Logic
   - The notion of Completeness
   - The Completeness Proof

2 Consequences of the Completeness theorem
   - Complexity of Validity
   - Compactness
   - Model Cardinality
   - Löwenheim-Skolem Theorem
   - Inexpressibility of Reachability
Outline

1. Completeness of proof system for First-Order Logic
   - The notion of Completeness
   - The Completeness Proof

2. Consequences of the Completeness theorem
   - Complexity of Validity
   - Compactness
   - Model Cardinality
   - Löwenheim-Skolem Theorem
   - Inexpressibility of Reachability
Soundness and Completeness

**Theorem**

*Soundness: If $\Delta \vdash \phi$, then $\Delta \models \phi$.***

**Theorem**

*Completeness (Gödel’s traditional form): If $\Delta \models \phi$, then $\Delta \vdash \phi$.***

**Theorem**

*Completeness (Gödel’s alternate form): If $\Delta$ is consistent, then it has a model.*
The traditional completeness theorem follows from the alternate form of the completeness theorem.

Proof.

Assume that $\Delta \models \phi$. It follows that any model $M$ that satisfies all the expressions in $\Delta$, also satisfies $\phi$ and hence falsifies $\neg \phi$. Thus, there does not exist a model that satisfies all the expressions in $\Delta \cup \{\neg \phi\}$. It follows that $\Delta \cup \{\neg \phi\}$ is inconsistent. But using the Contradiction theorem, it follows that $\Delta \vdash \phi$. 
Theorem

The traditional completeness theorem follows from the alternate form of the completeness theorem.

Proof.

Assume that $\Delta \models \phi$. It follows that any model $M$ that satisfies all the expressions in $\Delta$, also satisfies $\phi$ and hence falsifies $\neg \phi$. Thus, there does not exist a model that satisfies all the expressions in $\Delta \cup \{\neg \phi\}$. It follows that $\Delta \cup \{\neg \phi\}$ is inconsistent. But using the Contradiction theorem, it follows that $\Delta \nvdash \phi$. 
Theorem

The traditional completeness theorem follows from the alternate form of the completeness theorem.

Proof.

Assume that $\Delta \models \phi$. It follows that any model $M$ that satisfies all the expressions in $\Delta$, also satisfies $\phi$ and hence falsifies $\neg \phi$. Thus, there does not exist a model that satisfies all the expressions in $\Delta \cup \{\neg \phi\}$. It follows that $\Delta \cup \{\neg \phi\}$ is inconsistent. But using the Contradiction theorem, it follows that $\Delta \vdash \phi$. 
Theorem

The traditional completeness theorem follows from the alternate form of the completeness theorem.

Proof.

Assume that $\Delta \models \phi$. It follows that any model $M$ that satisfies all the expressions in $\Delta$, also satisfies $\phi$ and hence falsifies $\neg \phi$. Thus, there does not exist a model that satisfies all the expressions in $\Delta \cup \{\neg \phi\}$. It follows that $\Delta \cup \{\neg \phi\}$ is inconsistent. But using the Contradiction theorem, it follows that $\Delta \nvdash \phi$. 
Theorem

The traditional completeness theorem follows from the alternate form of the completeness theorem.

Proof.

Assume that $\Delta \models \phi$. It follows that any model $M$ that satisfies all the expressions in $\Delta$, also satisfies $\phi$ and hence falsifies $\neg \phi$. Thus, there does not exist a model that satisfies all the expressions in $\Delta \cup \{\neg \phi\}$. It follows that $\Delta \cup \{\neg \phi\}$ is inconsistent. But using the Contradiction theorem, it follows that $\Delta \not\models \phi$. 
Theorem

The traditional completeness theorem follows from the alternate form of the completeness theorem.

Proof.

Assume that $\Delta \models \phi$. It follows that any model $M$ that satisfies all the expressions in $\Delta$, also satisfies $\phi$ and hence falsifies $\neg \phi$. Thus, there does not exist a model that satisfies all the expressions in $\Delta \cup \{\neg \phi\}$. It follows that $\Delta \cup \{\neg \phi\}$ is inconsistent. But using the Contradiction theorem, it follows that $\Delta \vdash \phi$. 
Outline

1. Completeness of proof system for First-Order Logic
   - The notion of Completeness
   - The Completeness Proof

2. Consequences of the Completeness theorem
   - Complexity of Validity
   - Compactness
   - Model Cardinality
   - Löwenheim-Skolem Theorem
   - Inexpressibility of Reachability
Proof Sketch of Completeness Theorem

Proof.

http://www.maths.bris.ac.uk/~rp3959/firstordcomp.pdf
Outline

1. Completeness of proof system for First-Order Logic
   - The notion of Completeness
   - The Completeness Proof

2. Consequences of the Completeness theorem
   - Complexity of Validity
   - Compactness
   - Model Cardinality
   - Löwenheim-Skolem Theorem
   - Inexpressibility of Reachability
Validity

Theorem

Validity is recursively enumerable.

Proof.

Follows instantaneously from the completeness theorem.
Outline

1. Completeness of proof system for First-Order Logic
   - The notion of Completeness
   - The Completeness Proof

2. Consequences of the Completeness theorem
   - Complexity of Validity
   - Compactness
   - Model Cardinality
   - Löwenheim-Skolem Theorem
   - Inexpressibility of Reachability
Compactness

Theorem

If all finite subsets of a set of sentences $\Delta$ are satisfiable, then so is $\Delta$.

Proof.

Assume that $\Delta$ is unsatisfiable, but all finite subsets of $\Delta$ are satisfiable. As per the completeness theorem, there is a proof of a contradiction from $\Delta$, say $\Delta \vdash \phi \land \lnot \phi$. However, this proof has finite length! Therefore, it can involve only a finite subset of $\Delta$!
Compactness

Theorem

If all finite subsets of a set of sentences $\Delta$ are satisfiable, then so is $\Delta$.

Proof.

Assume that $\Delta$ is unsatisfiable, but all finite subsets of $\Delta$ are satisfiable. As per the completeness theorem, there is a proof of a contradiction from $\Delta$, say $\Delta \vdash \phi \land \neg \phi$. However, this proof has finite length! Therefore, it can involve only a finite subset of $\Delta$!
Compactness

Theorem

*If all finite subsets of a set of sentences $\Delta$ are satisfiable, then so is $\Delta$."

Proof.

Assume that $\Delta$ is unsatisfiable, but all finite subsets of $\Delta$ are satisfiable. As per the completeness theorem, there is a proof of a contradiction from $\Delta$, say $\Delta \vdash \phi \land \neg \phi$. However, this proof has finite length! Therefore, it can involve only a finite subset of $\Delta$!
Outline

1. Completeness of proof system for First-Order Logic
   - The notion of Completeness
   - The Completeness Proof

2. Consequences of the Completeness theorem
   - Complexity of Validity
   - Compactness
   - Model Cardinality
   - Löwenheim-Skolem Theorem
   - Inexpressibility of Reachability
Theorem

If a sentence has a model, it has a countable model.

Proof.

The model $M$ constructed in the proof of the completeness theorem is countable, since the corresponding vocabulary is countable.
Outline

1. Completeness of proof system for First-Order Logic
   - The notion of Completeness
   - The Completeness Proof

2. Consequences of the Completeness theorem
   - Complexity of Validity
   - Compactness
   - Model Cardinality
   - Löwenheim-Skolem Theorem
   - Inexpressibility of Reachability
Query

Do all sentences have infinite models?

Theorem

If a sentence \( \phi \) has finite models of arbitrary large cardinality, then it has an infinite model.

Proof.

Consider the sentence \( \psi_k = \exists x_1 \exists x_2 \ldots \exists x_k \land \land_{1 \leq i < j \leq k} \neg (x_i = x_j) \). \( \psi_k \) cannot be satisfied with a model having less than \( k \) elements.

Assume that \( \phi \) has arbitrarily large models, but no infinite models. Let \( \Delta = \phi \cup \{ \psi_k \mid k = 2, 3, \ldots \} \). If \( \Delta \) has a model \( M \), \( M \) can neither be finite nor infinite. Thus, \( \Delta \) does not have a model. By the compactness theorem, a finite subset \( D \subseteq \Delta \) does not have a model. \( \phi \) must be in \( D \). Let \( k \) denote the largest integer, such that \( \psi_k \in D \). But there is a large enough model that satisfies both \( \phi \) (hypothesis) and \( \psi_k \).
Query

Do all sentences have infinite models?

Theorem

If a sentence \( \phi \) has finite models of arbitrary large cardinality, then it has an infinite model.

Proof.

Consider the sentence \( \psi_k = \exists x_1 \exists x_2 \ldots \exists x_k \land 1 \leq i < j \leq k \neg(x_i = x_j) \). \( \psi_k \) cannot be satisfied with a model having less than \( k \) elements.

Assume that \( \phi \) has arbitrarily large models, but no infinite models. Let 
\[ \Delta = \phi \cup \{\psi_k : k = 2, 3, \ldots\} \]. If \( \Delta \) has a model \( M \), \( M \) can neither be finite nor infinite. Thus, \( \Delta \) does not have a model. By the compactness theorem, a finite subset \( D \subset \Delta \) does not have a model. \( \phi \) must be in \( D \). Let \( k \) denote the largest integer, such that \( \psi_k \in D \). But there is a large enough model that satisfies both \( \phi \) (hypothesis) and \( \psi_k \)!
Query

Do all sentences have infinite models?

Theorem

If a sentence $\phi$ has finite models of arbitrary large cardinality, then it has an infinite model.

Proof.

Consider the sentence $\psi_k = \exists x_1 \exists x_2 \ldots \exists x_k \land_{1 \leq i < j \leq k} \neg (x_i = x_j)$. $\psi_k$ cannot be satisfied with a model having less than $k$ elements.

Assume that $\phi$ has arbitrarily large models, but no infinite models. Let $\Delta = \phi \cup \{\psi_k : k = 2, 3, \ldots\}$. If $\Delta$ has a model $M$, $M$ can neither be finite nor infinite. Thus, $\Delta$ does not have a model. By the compactness theorem, a finite subset $D \subseteq \Delta$ does not have a model. $\phi$ must be in $D$. Let $k$ denote the largest integer, such that $\psi_k \in D$. But there is a large enough model that satisfies both $\phi$ (hypothesis) and $\psi_k$!
Query

Do all sentences have infinite models?

Theorem

If a sentence $\phi$ has finite models of arbitrary large cardinality, then it has an infinite model.

Proof.

Consider the sentence $\psi_k = \exists x_1 \exists x_2 \ldots \exists x_k \land 1 \leq i < j \leq k \neg(x_i = x_j)$. $\psi_k$ cannot be satisfied with a model having less than $k$ elements.

Assume that $\phi$ has arbitrarily large models, but no infinite models. Let $\Delta = \phi \cup \{\psi_k : k = 2, 3, \ldots\}$. If $\Delta$ has a model $M$, $M$ can neither be finite nor infinite. Thus, $\Delta$ does not have a model. By the compactness theorem, a finite subset $D \subset \Delta$ does not have a model. $\phi$ must be in $D$. Let $k$ denote the largest integer, such that $\psi_k \in D$. But there is a large enough model that satisfies both $\phi$ (hypothesis) and $\psi_k$!
**Query**

Do all sentences have infinite models?

**Theorem**

If a sentence $\phi$ has finite models of arbitrary large cardinality, then it has an infinite model.

**Proof.**

Consider the sentence $\psi_k = \exists x_1 \exists x_2 \ldots \exists x_k \land_{1 \leq i < j \leq k} \neg (x_i = x_j)$. $\psi_k$ cannot be satisfied with a model having less than $k$ elements.

Assume that $\phi$ has arbitrarily large models, but no infinite models. Let $\Delta = \phi \cup \{\psi_k : k = 2, 3, \ldots\}$. If $\Delta$ has a model $M$, $M$ can neither be finite nor infinite. Thus, $\Delta$ does not have a model. By the compactness theorem, a finite subset $D \subset \Delta$ does not have a model. $\phi$ must be in $D$. Let $k$ denote the largest integer, such that $\psi_k \in D$. But there is a large enough model that satisfies both $\phi$ (hypothesis) and $\psi_k$!
1. Completeness of proof system for First-Order Logic
   - The notion of Completeness
   - The Completeness Proof

2. Consequences of the Completeness theorem
   - Complexity of Validity
   - Compactness
   - Model Cardinality
   - Löwenheim-Skolem Theorem
   - Inexpressibility of Reachability
REACHABILITY

Given a directed graph $G$ and two nodes $x$ and $y$ in $G$, is there a directed path from $x$ to $y$ in $G$?

Theorem

There is no first-order expression $\phi$ with two free variables $x$ and $y$, such that $\phi$-Graphs expresses REACHABILITY.

Proof.

Assume that there exists such a $\phi$. Consider the sentence, $\psi' = \psi_0 \land \psi_1 \land \psi_2$, where,

\[
\begin{align*}
\psi_0 &= (\forall x)(\forall y)\phi \\
\psi_1 &= (\forall x)(\exists y)G(x, y) \land (\forall x)(\forall y)(\forall z)((G(x, y) \land G(x, z)) \rightarrow (y = z)) \\
\psi_2 &= (\forall x)(\exists y)G(y, x) \land (\forall x)(\forall y)(\forall z)((G(y, x) \land G(z, x)) \rightarrow (y = z))
\end{align*}
\]

 Arbitrarily large models are possible for $\psi'$, but no infinite models!
REACHABILITY

Given a directed graph $G$ and two nodes $x$ and $y$ in $G$, is there a directed path from $x$ to $y$ in $G$?

Theorem

There is no first-order expression $\phi$ with two free variables $x$ and $y$, such that $\phi$-Graphs expresses REACHABILITY.

Proof.

Assume that there exists such a $\phi$. Consider the sentence, $\psi' = \psi_0 \land \psi_1 \land \psi_2$, where,

$$
\psi_0 = (\forall x)(\forall y)\phi
$$

$$
\psi_1 = (\forall x)(\exists y)G(x, y) \land (\forall x)(\forall y)(\forall z)((G(x, y) \land G(x, z)) \rightarrow (y = z))
$$

$$
\psi_2 = (\forall x)(\exists y)G(y, x) \land (\forall x)(\forall y)(\forall z)((G(y, x) \land G(z, x)) \rightarrow (y = z))
$$

Arbitrarily large models are possible for $\psi'$, but no infinite models!
Reachability

Given a directed graph $G$ and two nodes $x$ and $y$ in $G$, is there a directed path from $x$ to $y$ in $G$?

Theorem

There is no first-order expression $\phi$ with two free variables $x$ and $y$, such that $\phi$-Graphs expresses Reachability.

Proof.

Assume that there exists such a $\phi$. Consider the sentence, $\psi' = \psi_0 \land \psi_1 \land \psi_2$, where,

\[
\begin{align*}
\psi_0 & = (\forall x)(\forall y) \phi \\
\psi_1 & = (\forall x)(\exists y) G(x, y) \land (\forall x)(\forall y)(\forall z)((G(x, y) \land G(x, z)) \rightarrow (y = z)) \\
\psi_2 & = (\forall x)(\exists y) G(y, x) \land (\forall x)(\forall y)(\forall z)((G(y, x) \land G(z, x)) \rightarrow (y = z))
\end{align*}
\]

Arbitrarily large models are possible for $\psi'$, but no infinite models!
Reachability

Given a directed graph \( G \) and two nodes \( x \) and \( y \) in \( G \), is there a directed path from \( x \) to \( y \) in \( G \)?

Theorem

*There is no first-order expression \( \phi \) with two free variables \( x \) and \( y \), such that \( \phi \)-Graphs expresses Reachability.*

Proof.

Assume that there exists such a \( \phi \). Consider the sentence, \( \psi' = \psi_0 \land \psi_1 \land \psi_2 \), where,

\[
\begin{align*}
\psi_0 &= (\forall x)(\forall y)\phi \\
\psi_1 &= (\forall x)(\exists y)G(x, y) \land (\forall x)(\forall y)(\forall z)((G(x, y) \land G(x, z)) \rightarrow (y = z)) \\
\psi_2 &= (\forall x)(\exists y)G(y, x) \land (\forall x)(\forall y)(\forall z)((G(y, x) \land G(z, x)) \rightarrow (y = z))
\end{align*}
\]

Arbitrarily large models are possible for \( \psi' \), but no infinite models!