Undecidability in Logic - Part I

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Number Theory and Computation
1. Axiomatizing Number Theory
   - Non-logical Axioms
   - Sample Proof
   - Complete fragments of number theory

2. Complexity as a number-theoretic concept
   - Representing Turing Machines as numbers
   - Encoding sample
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Outline

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A set of Axioms

Non-logical Axioms

\(\text{NT1} \quad (\forall x)(\sigma(x) \neq 0).\)
\(\text{NT2} \quad (\forall x)(\forall y)[(\sigma(x) = \sigma(y)) \rightarrow (x = y)].\)
\(\text{NT3} \quad (\forall x)((x = 0) \vee (\exists y)(x = \sigma(y))).\)
\(\text{NT4} \quad (\forall x)(x + 0 = x).\)
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A set of Axioms (contd.)

Notational convenience

(i) \( \mod(x, y, z) \) is an abbreviation for \( (\exists w)((x = y \times w + z) \land (z < y)) \).
(ii) \( \div(x, y, w) \) is an abbreviation for \( (\exists z)((x = y \times w + z) \land (z < y)) \).
(iii) \( \text{NT} = \text{NT}_1 \land \text{NT}_2 \land \ldots \land \text{NT}_{14} \)
(iv) We use 1 for \( \sigma(0) \), 2 for \( \sigma(\sigma(0)) \), 3 for \( \sigma(\sigma(\sigma(0))) \) and so on.

Properties of Axiom set

(i) Is it sound? Yes! If \( \text{NT} \vdash \phi \), then \( \text{N} \models \phi \). Use induction on the number of steps in the proof sequence of \( \text{NT} \vdash \phi \).
(ii) Is it complete? i.e., if \( \text{N} \models \phi \), does \( \text{NT} \vdash \phi \)? Apparently not! For instance, there is no proof from \( \text{NT} \) of the valid sentence \( (\forall x)(\forall y)[(x + y) = (y + x)] \). In fact, no system of axioms exists for \( \text{N} \), that is both sound and complete.
Notational convenience

(i) \( \text{mod}(x, y, z) \) is an abbreviation for \( \exists w \left( (x = y \times w + z) \land (z < y) \right) \).

(ii) \( \text{div}(x, y, w) \) is an abbreviation for \( \exists z \left( (x = y \times w + z) \land (z < y) \right) \).

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(i) Is it sound? Yes! If \( \text{NT} \vdash \phi \), then \( \text{N} \models \phi \). Use induction on the number of steps in the proof sequence of \( \text{NT} \vdash \phi \).

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A set of Axioms (contd.)

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(i) \( \text{mod}(x, y, z) \) is an abbreviation for \( (\exists w)((x = y \times w + z) \land (z < y)) \).
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Outline

1. Axiomatizing Number Theory
   - Non-logical Axioms
   - Sample Proof
   - Complete fragments of number theory

2. Complexity as a number-theoretic concept
   - Representing Turing Machines as numbers
   - Encoding sample
Sample Proof

Example

Show that \( \text{NT} \vdash 1 < 1 + 1 \).

Proof.

Consider the following proof sequence:

(i) \((\forall x)(\forall y)((x + \sigma(y)) = \sigma(x + y)), \text{ NT5}.\)
(ii) \((\forall x)((x + \sigma(0)) = \sigma(x + 0)), \text{ (i), u.i. (setting } y = 0).\)
(iii) \((\forall x)((x + 1) = \sigma(x)), \text{ NT4}.\)
(iv) \((\forall x)(\sigma(x) = x + 1), \text{ properties of equality}.\)
(v) \((\forall x)(x < \sigma(x)), \text{ NT10}.\)
(vi) \(1 < \sigma(1), \text{ (v), u.i. (setting } x = 1).\)
(vii) \(\sigma(1) = 1 + 1, \text{ (iv), u.i. (setting } x = 1).\)
(viii) \(1 < 1 + 1, \text{ (vi), (vii)}.\)
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Variable-Free Sentences

**Theorem**

If \( \phi \) is a variable-free sentence, then \( \mathbb{N} \models \phi \iff \text{NT} \vdash \phi \).

**Proof.**

Any variable-free sentence is an arbitrary boolean combination of expressions of the form: \( t = t' \) and \( t < t' \).

(i) \( t \) and \( t' \) are numbers - \( t = t' \) is trivial to prove. \( t < t' \) can be proved by using \( \text{NT10} \) to prove \( t < \sigma(t), \sigma(t) < \sigma(\sigma(t)) \) and so on. Eventually, we can use \( \text{NT13} \) to establish the inequality.

(ii) \( t \) and \( t' \) are general variable-free terms (e.g., \( t = 2 \uparrow 3 + (4 \times 7) + 6 \)) - Both \( t \) and \( t' \) have values, say \( t_0 \) and \( t'_0 \). We need to show that \( \text{NT} \vdash t = t_0 \) and \( \text{NT} \vdash t' = t'_0 \). Use induction on structure of \( t \), by repeatedly applying the axioms \( \text{NT9, NT7 and NT5} \). Ultimately, the expression will be reduced to its value.
Variable-Free Sentences

**Theorem**

*If φ is a variable-free sentence, then N \models φ ⇔ NT ⊢ φ.*

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Bounded Quantifiers

Notation

(i) $(\forall x < t)\phi$ stands for $(\forall x)((x < t) \rightarrow \phi)$. Bounded prenex form.
(ii) Bounded sentence.

Theorem

Suppose that $\phi$ is a bounded sentence. Then $\mathbf{N} \models \phi \iff \mathbf{NT} \vdash \phi$.

Proof.

Since $\mathbf{NT}$ is sound, $\mathbf{NT} \vdash \phi \rightarrow \mathbf{N} \models \phi$. We use induction on the number of quantifiers to prove the converse.

(i) $\phi$ has no quantifiers - Variable-Free sentence!

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Undecidability in Logic
Bounded Quantifiers

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(i) \((\forall x < t) \phi\) stands for \((\forall x)((x < t) \rightarrow \phi)\). Bounded prenex form.

(ii) Bounded sentence.

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Suppose that \(\phi\) is a bounded sentence. Then \(N \models \phi \iff NT \vdash \phi\).

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Since \(NT\) is sound, \(NT \vdash \phi \implies N \models \phi\). We use induction on the number of quantifiers to prove the converse.

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1. Axiomatizing Number Theory
   - Non-logical Axioms
   - Sample Proof
   - Complete fragments of number theory

2. Complexity as a number-theoretic concept
   - Representing Turing Machines as numbers
   - Encoding sample
Axiomatizing Number Theory
Complexity as a number-theoretic concept

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Procedure

Let $M = (K, \Sigma, \delta, s)$ denote a Turing Machine.

(i) Represent the symbols in $\Sigma$ using integers in $\{0, 1, \ldots, |\Sigma| - 1\}$ and the symbols in $K$ using integers in $\{\Sigma, \Sigma+1, \ldots, |\Sigma| + |K| - 1\}$.

(ii) $s$ is always encoded as $|\Sigma|$ and 0 is always used to encode $\triangleright$.

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(iv) $\sqcup$ is encoded by 1.

Thus, all symbols can be encoded using $b = |\Sigma| + |K|$ integers. Consider the configuration $C = (q, w, u)$, where $q \in K$ and $w = w_1, w_2, \ldots, w_m$ and $u = u_1, u_2, \ldots u_n \in \Sigma^*$. $C$ can be thought of as the unique integer whose $b$-ary representation is

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<tr>
<th>$p \in K$, $\sigma \in \Sigma$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$s$</td>
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Table: A Turing Machine that accepts $a^*$

Characteristics

$|K| = |\Sigma| = 4$ and hence $b = 8$.

The configuration $(q, \triangleright aa, \sqcup \sqcup)$ is represented by the sequence $(0, 2, 2, 7, 1, 1)$ or by the integer $022711_8$ or $9673_{10}$.

Observation

The relation “yields in one step” over the configurations of $M$ defines a relation $Y_M \subseteq N^2$.

Goal

To formulate a first-order expression $\text{yields}_M(x, y)$ in number theory, over the free variables $x$ and $y$, such that

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\hline
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s & b & (s, b, \rightarrow) \\
s & \sqcup & (q, \sqcup, \leftarrow) \\
s & \triangleright & (q, \triangleright, \rightarrow) \\
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<td>$s$ $b$</td>
<td>$(s, b, \rightarrow)$</td>
</tr>
<tr>
<td>$s$ $\square$</td>
<td>$(q, \square, \leftarrow)$</td>
</tr>
<tr>
<td>$s$ $\triangleright$</td>
<td>$(q, \triangleright, \rightarrow)$</td>
</tr>
<tr>
<td>$q$ $a$</td>
<td>$(q, \square, \leftarrow)$</td>
</tr>
<tr>
<td>$q$ $b$</td>
<td>(&quot;no&quot;, $b$, $-$)</td>
</tr>
<tr>
<td>$q$ $\triangleright$</td>
<td>(&quot;yes&quot;, $\triangleright$, $\rightarrow$)</td>
</tr>
</tbody>
</table>

Observation

The relation "yields in one step" over the configurations of $M$ defines a relation $Y_M \subseteq \mathbb{N}^2$.

Goal

To formulate a first-order expression $\text{yields}_M(x, y)$ in number theory, over the free variables $x$ and $y$, such that

$\mathbb{N}_{x=m, y=n} \models \text{yields}_M(x, y)$ iff $Y_M(m, n)$.

Characteristics

$|K| = |\Sigma| = 4$ and hence $b = 8$.

The configuration $(q, \triangleright aa, \square \square)$ is represented by the sequence $(0, 2, 2, 7, 1, 1)$ or by the integer $022711_8$ or $9673_{10}$. 

Subramani

Undecidability in Logic