A sampling of Randomized Algorithms

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Main points

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Main points

Probability spaces, Random Variable, Distribution of a random variable (pmf), Expected Value,
Recap

Main points

Probability spaces, Random Variable, Distribution of a random variable (pmf), Expected Value, Variance.
Polynomial Identities and the verification problem
A function \( f() \) of one variable \( x \), is called a polynomial function, if it satisfies,

\[
f(x) = \sum_{i=0}^{n} a_{n-i} \cdot x^{n-i}.
\]
**Definition**

A function $f()$ of one variable $x$, is called a polynomial function, if it satisfies,

$$f(x) = \sum_{i=0}^{n} a_{n-i} \cdot x^{n-i}.$$  

The above form of $f()$ is called the *canonical form*.
Polynomial Identities and the verification problem

**Definition**

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The above form of \( f() \) is called the **canonical form**.

**Note**

A univariate polynomial function can also be expressed in the form:

\[
f(x) = \prod_{i=1}^{n} (a_i x - b_i)
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Definition

A function \( f() \) of one variable \( x \), is called a polynomial function, if it satisfies,

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f(x) = \sum_{i=0}^{n} a_{n-i} \cdot x^{n-i}.
\]

The above form of \( f() \) is called the \textit{canonical form}.

Note

A \textit{univariate polynomial function} can also be expressed \textit{in the form}:

\[
f(x) = \prod_{i=1}^{n} (a_{i}x - b_{i})
\]

This form is called the \textit{product form}.
Polynomial Representation

Representation
Polynomial Representation
Polynomial Representation

(i) Without loss of generality we will assume that the coefficient of $x^n$ is 1 in the polynomials that we consider (both canonical form and product forms).
Polynomial Representation

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(ii) Accordingly, a polynomial function in the canonical form can be written as:

$$f(x) = x^n + a_{n-1} \cdot x^{n-1} + a_{n-2} \cdot x^{n-2} + \ldots a_0$$
Without loss of generality we will assume that the coefficient of $x^n$ is 1 in the polynomials that we consider (both canonical form and product forms).

Accordingly, a polynomial function in the canonical form can be written as:

$$f(x) = x^n + a_{n-1} \cdot x^{n-1} + a_{n-2} \cdot x^{n-2} + \ldots a_0$$

and a polynomial function in product form can be written as:

$$g(x) = \prod_{i=1}^n (x_i - c_i)$$
Statement of Problem
Statement of Problem

Problem

Given two polynomial functions $f()$ and $g()$ in canonical form and product form respectively, is $f() = g()$?
Statement of Problem

Problem

Given two polynomial functions $f()$ and $g()$ in canonical form and product form respectively, is $f() = g()$?

Theorem

*Every polynomial function has a unique canonical representation.*
Deterministic Approach
Deterministic Approach

**Function** VERIFY-POLYNOMIAL-IDENTITY(f(), g())

1. Convert $g()$ into canonical form.
**Deterministic Approach**

**Function**  \textsc{Verify-Polynomial-Identity}(f(), g())

1. Convert $g()$ into canonical form.
2. Check if the coefficients of $g()$ match up with the coefficients of $f()$. 
Deterministic Approach

**Function** $\text{VERIFY-POLYNOMIAL-IDENTITY}(f(), g())$

1. Convert $g()$ into canonical form.
2. Check if the coefficients of $g()$ match up with the coefficients of $f()$. 

**Analysis**
Deterministic Approach

Function `VERIFY-POLYNOMIAL-IDENTITY(f(), g())`

1. Convert `g()` into canonical form.
2. Check if the coefficients of `g()` match up with the coefficients of `f()`.

Analysis

What is the running time?
Deterministic Approach

Function **VERIFY-POLYNOMIAL-IDENTITY**($f()$, $g()$)

1. Convert $g()$ into canonical form.
2. Check if the coefficients of $g()$ match up with the coefficients of $f()$.

Analysis

What is the running time? $\Theta(n^2)$!
Deterministic Approach

**Function** \( \text{VERIFY-POLYNOMIAL-IDENTITY}(f(), g()) \)

1. Convert \( g() \) into canonical form.
2. Check if the coefficients of \( g() \) match up with the coefficients of \( f() \).

**Analysis**

What is the running time? \( \Theta(n^2) \)! Can we do better?
A randomized approach
A randomized approach

Randomized Approach

Function \textsc{Randomized-Verify-Polynomial-Identity}(f(), g())

1: Pick an integer \( r \) uniformly from the interval \( \{1, 2, \ldots, 2 \cdot n\} \).
2: Compute \( s = f(r) \) and \( t = g(r) \).
3: \textbf{if} \( s = t \) \textbf{then}
4: \( f() \) and \( g() \) are identical.
5: \textbf{else}
6: \( f() \) and \( g() \) are \textbf{not} identical.
7: \textbf{end if}
Recap

Verifying polynomial Identities
Verifying Matrix Multiplication
A Randomized Min-Cut Algorithm
Coupon Collector problems

A randomized approach

Randomized Approach

Function `RANDOMIZED-VERIFY-POLYNOMIAL-IDENTITY(f(), g())`

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Running Time
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Running Time

\( O(n) \) for computing \( t \). How much time for computing \( s \)?
A randomized approach

Randomized Approach

**Function** `RANDOMIZED-VERIFY-POLYNOMIAL-IDENTITY(f(), g())`

1. Pick an integer `r` uniformly from the interval `{1, 2, ..., 2*n}`.
2. Compute `s = f(r)` and `t = g(r)`.
3. if `(s = t)` then
4.   `f()` and `g()` are identical.
5. else
6.   `f()` and `g()` are **not** identical.
7. end if

Running Time

`O(n)` for computing `t`. How much time for computing `s`? `O(n^2)`?
A randomized approach

**Randomized Approach**

**Function** `RANDOMIZED-VERIFY-POLYNOMIAL-IDENTITY(f(), g())`

1. Pick an integer $r$ uniformly from the interval $\{1, 2, \ldots, 2 \cdot n\}$.
2. Compute $s = f(r)$ and $t = g(r)$.
3. **if** ($s = t$) **then**
4. $f()$ and $g()$ are identical.
5. **else**
6. $f()$ and $g()$ are **not** identical.
7. **end if**

**Running Time**

$O(n)$ for computing $t$. How much time for computing $s$? $O(n^2)$? Actually $\Theta(n)$!
Correctness analysis

Correctness
Correctness analysis

Correctness

(i) When $f() = g()$, does the randomized algorithm make a mistake?
Correctness analysis

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(i) When $f() = g()$, does the randomized algorithm make a mistake?
(ii) When $f() \neq g()$, does the randomized algorithm make a mistake?
Correctness analysis

Correctness

(i) When $f() = g()$, does the randomized algorithm make a mistake?

(ii) When $f() \neq g()$, does the randomized algorithm make a mistake?

(iii) What can you conclude when the randomized algorithm declares that $f() \neq g()$?
Correctness analysis

Correctness

(i) When $f() = g()$, does the randomized algorithm make a mistake?
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Correctness analysis

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(i) When $f() = g()$, does the randomized algorithm make a mistake?
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(iv) What can you conclude when the randomized algorithm declares that $f() = g()$?

Note

*If the algorithm declares that $f() \neq g()$, then the algorithm is correct.*
**Correctness analysis**

<table>
<thead>
<tr>
<th>Correctness</th>
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<tbody>
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**Note**

*If the algorithm declares that \( f() \neq g() \), then the algorithm is correct. If the algorithm declares that \( f() = g() \), then it is possible that \( f() \neq g() \).*
Correctness analysis

Correctness

(i) When \( f() = g() \), does the randomized algorithm make a mistake?
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(iv) What can you conclude when the randomized algorithm declares that \( f() = g() \)?

Note

If the algorithm declares that \( f() \neq g() \), then the algorithm is correct. If the algorithm declares that \( f() = g() \), then it is possible that \( f() \neq g() \). We need to bound the probability of this event.
Bounding the error probability
Bounding the error probability

Definition

The values at which a polynomial function evaluates to zero are called its roots.
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Theorem

A polynomial of degree n has exactly n roots
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A polynomial of degree $n$ has exactly $n$ roots (not necessarily distinct).
Definition

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Theorem

A polynomial of degree $n$ has exactly $n$ roots (not necessarily distinct). (Fundamental theorem of algebra).

Observation

Two distinct polynomials $f()$ and $g()$ can be equal only at the roots of the polynomial $f() – g()$. 
Definition

The values at which a polynomial function evaluates to zero are called its roots.

Theorem

A polynomial of degree $n$ has exactly $n$ roots (not necessarily distinct). (Fundamental theorem of algebra).

Observation

Two distinct polynomials $f()$ and $g()$ can be equal only at the roots of the polynomial $f() - g()$. The polynomial $f() - g()$ has at most $n$ distinct roots.
Bounding the error probability (contd.)
Observation

The only way for the randomized algorithm to give an incorrect answer when \( f() \neq g() \), is if the integer \( r \) that it picked, is a root of the polynomial \( f() - g() \).
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The only way for the randomized algorithm to give an incorrect answer when \( f() \neq g() \), is if the integer \( r \) that it picked, is a root of the polynomial \( f() - g() \). There can be at most \( n \) roots of \( f() - g() \) in the range \( \{1, 2, \ldots, 2 \cdot n\} \). Since \( r \) is chosen uniformly and at random,
Observation

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Observation

The only way for the randomized algorithm to give an incorrect answer when \( f(\) \( \neq g(\), is if the integer \( r \) that it picked, is a root of the polynomial \( f(\) – \( g(\). There can be at most \( n \) roots of \( f(\) – \( g(\) in the range \( \{1, 2, \ldots, 2 \cdot n\} \). Since \( r \) is chosen uniformly and at random, the probability that \( r \) is a root of \( f(\) – \( g(\) is at most \( \frac{n}{2 \cdot n} = \frac{1}{2} \).

Theorem

On “yes” instances, the randomized algorithm does not err. On “no” instances, the probability that the algorithm errs is at most \( \frac{1}{2} \).
Matrix multiplication and Verification
Problem Statement

Given 2 square $n \times n$ matrices $A$ and $B$, compute $C = A \cdot B$. 
Matrix multiplication and Verification

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We will assume that all entries in $A$ and $B$ belong to the set $\{0, 1\}$. 
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Deterministic Approaches
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Deterministic Approaches
(i) Naive.
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(i) Naive.
(ii) Strassen.
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Deterministic Approaches
(i) Naive.
(ii) Strassen.

How to verify Strassen?
A randomized approach to verification

Randomized Approach

**Function** RANDOMIZED-VERIFY-MATRIX-PRODUCT(C, A, B)

1: Pick a vector $r$ uniformly from the box $\{0, 1\}^n$.
2: Compute $s = A \cdot B \cdot r$ and $t = C \cdot r$.
3: if $(s = t)$ then
4:   $C$ is the product of $A$ and $B$.
5: else
6:   $C$ is not the product of $A$ and $B$.
7: end if
A randomized approach to verification

Randomized Approach

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5: else
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7: end if

Running Time
A randomized approach to verification

Randomized Approach

Function \textsc{Randomized-Verify-Matrix-Product}(C, A, B)

1: Pick a vector \( r \) uniformly from the box \( \{0, 1\}^n \).
2: Compute \( s = A \cdot B \cdot r \) and \( t = C \cdot r \).
3: \textbf{if} \( s = t \) \textbf{then}
4: \hspace{1em} \( C \) is the product of \( A \) and \( B \).
5: \textbf{else}
6: \hspace{1em} \( C \) is \textbf{not} the product of \( A \) and \( B \).
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Running Time

\( O(n^2) \) for computing \( t \). How much time for computing \( s \)?
A randomized approach to verification

Randomized Approach

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Running Time

$O(n^2)$ for computing $t$. How much time for computing $s$? $O(n^3 + n^2 = n^3)$?
A randomized approach to verification

Randomized Approach

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1: Pick a vector $r$ uniformly from the box $\{0, 1\}^n$.
2: Compute $s = A \cdot B \cdot r$ and $t = C \cdot r$.
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Running Time

$O(n^2)$ for computing $t$. How much time for computing $s$? $O(n^3 + n^2 = n^3)$? Actually $O(n^2)$!
Observation

If $A \cdot B = C$, then the randomized algorithm does not err.
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If $A \cdot B = C$, then the randomized algorithm does not err. If $A \cdot B \neq C$, then the randomized algorithm could err, since it could be the case that $A \cdot B \cdot r = C \cdot r$, for the chosen $r$. 


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If $A \cdot B = C$, then the randomized algorithm does not err. If $A \cdot B \neq C$, then the randomized algorithm could err, since it could be the case that $A \cdot B \cdot r = C \cdot r$, for the chosen $r$, but there exists some other vector $u$, such that $A \cdot B \cdot u \neq C \cdot u$. 
Observation

If $A \cdot B = C$, then the randomized algorithm does not err. If $A \cdot B \neq C$, then the randomized algorithm could err, since it could be the case that $A \cdot B \cdot r = C \cdot r$, for the chosen $r$, but there exists some other vector $u$, such that $A \cdot B \cdot u \neq C \cdot u$. We need to bound the probability of the error, over the random choices made by the algorithm.
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If \( A \cdot B = C \), then the randomized algorithm does not err. If \( A \cdot B \neq C \), then the randomized algorithm could err, since it could be the case that \( A \cdot B \cdot r = C \cdot r \), for the chosen \( r \), but there exists some other vector \( u \), such that \( A \cdot B \cdot u \neq C \cdot u \). We need to bound the probability of the error, over the random choices made by the algorithm.

Main idea

We will show that if \( r \) is chosen uniformly from \( \{0,1\}^n \), then the probability that \( A \cdot B \cdot r = C \cdot r \), when \( A \cdot B \neq C \) is at most \( \frac{1}{2} \).
Observation

If $A \cdot B = C$, then the randomized algorithm does not err. If $A \cdot B \neq C$, then the randomized algorithm could err, since it could be the case that $A \cdot B \cdot r = C \cdot r$, for the chosen $r$, but there exists some other vector $u$, such that $A \cdot B \cdot u \neq C \cdot u$. We need to bound the probability of the error, over the random choices made by the algorithm.

Main idea

We will show that if $r$ is chosen uniformly from $\{0,1\}^n$, then the probability that $A \cdot B \cdot r = C \cdot r$, when $A \cdot B \neq C$ is at most $\frac{1}{2}$.

Lemma

There is no difference between choosing $r$ uniformly from $\{0,1\}^n$ and choosing each of its components uniformly over the set $\{0,1\}$.
Analysis (contd.)
Analysis (contd.)

Bounding the error
Assume that $A \cdot B \neq C$, but that $A \cdot B \cdot r = C \cdot r$. 
### Bounding the error

(i) Assume that $A \cdot B \neq C$, but that $A \cdot B \cdot r = C \cdot r$.

(ii) Let $D = A \cdot B - C$. 

Bounding the error

(i) Assume that $A \cdot B \neq C$, but that $A \cdot B \cdot r = C \cdot r$.

(ii) Let $D = A \cdot B - C$. Can $D$ be $0$?
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(i) Assume that $A \cdot B \neq C$, but that $A \cdot B \cdot r = C \cdot r$.

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(iii) Without loss of generality assume that the first element of the first row of $D$, i.e., $d_{11}$ is not 0.
Bounding the error

(i) Assume that $\mathbf{A} \cdot \mathbf{B} \neq \mathbf{C}$, but that $\mathbf{A} \cdot \mathbf{B} \cdot \mathbf{r} = \mathbf{C} \cdot \mathbf{r}$.

(ii) Let $\mathbf{D} = \mathbf{A} \cdot \mathbf{B} - \mathbf{C}$. Can $\mathbf{D}$ be $\mathbf{0}$?

(iii) Without loss of generality assume that the first element of the first row of $\mathbf{D}$, i.e., $d_{11}$ is not 0.

(iv) Since $\mathbf{D} \cdot \mathbf{r} = 0$, it means that
Bounding the error

(i) Assume that $A \cdot B \neq C$, but that $A \cdot B \cdot r = C \cdot r$.

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(iv) Since $D \cdot r = 0$, it means that $\sum_{i=1}^{n} d_{1i} \cdot r_i = 0$. 
Bounding the error

(i) Assume that \( A \cdot B \neq C \), but that \( A \cdot B \cdot r = C \cdot r \).

(ii) Let \( D = A \cdot B - C \). Can \( D \) be 0?

(iii) Without loss of generality assume that the first element of the first row of \( D \), i.e., \( d_{11} \) is not 0.

(iv) Since \( D \cdot r = 0 \), it means that \( \sum_{i=1}^{n} d_{1i} \cdot r_i = 0 \).

(v) Therefore, \( r_1 = -\frac{\sum_{j=2}^{n} d_{ij} \cdot r_j}{d_{11}} \).
Analysis (contd.)

Bounding the error

(i) Assume that $A \cdot B \neq C$, but that $A \cdot B \cdot r = C \cdot r$.

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(v) Therefore, $r_{1} = -\frac{\sum_{j=2}^{n} d_{ij} \cdot r_{j}}{d_{11}}$.

(vi) Assume that $r_{2}, r_{3}, \ldots, r_{n}$ are chosen before $r_{1}$. 
Bounding the error

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(v) Therefore, $r_{1} = -\frac{\sum_{j=2}^{n} d_{ij} \cdot r_{j}}{d_{11}}$.

(vi) Assume that $r_{2}, r_{3}, \ldots, r_{n}$ are chosen before $r_{1}$. Thus the RHS is fixed and evaluated before $r_{1}$ is chosen.
Bounding the error

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(vi) Assume that $r_2, r_3, \ldots, r_n$ are chosen before $r_1$. Thus the RHS is fixed and evaluated before $r_1$ is chosen. Thus the probability that the value of $r_1$ chosen will be equal to the RHS value is at most $\frac{1}{2}$.
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(v) Therefore, $r_1 = -\frac{\sum_{i=2}^{n} d_{ij} \cdot r_j}{d_{11}}$.

(vi) Assume that $r_2, r_3, \ldots, r_n$ are chosen before $r_1$. Thus the RHS is fixed and evaluated before $r_1$ is chosen. Thus the probability that the value of $r_1$ chosen will be equal to the RHS value is at most $\frac{1}{2}$.

Note

The above method of analysis is called the Principle of Deferred Decisions.
The Min-cut problem on undirected unweighted graphs
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Definition

Given an undirected graph $G = \langle V, E \rangle$, 

The Min-cut problem on undirected unweighted graphs

**Definition**

Given an undirected graph $G = \langle V, E \rangle$, a *cut-set* or *cut* is defined as some set of edges $E' \subseteq E$, 
The Min-cut problem on undirected unweighted graphs

**Definition**

Given an undirected graph $G = \langle V, E \rangle$, a *cut-set* or *cut* is defined as some set of edges $E' \subseteq E$, whose removal disconnects $G$. 
The Min-cut problem on undirected unweighted graphs

**Definition**

Given an undirected graph $G = \langle V, E \rangle$, a *cut-set* or *cut* is defined as some set of edges $E' \subseteq E$, whose removal disconnects $G$. In other words, $G = \langle V, E - E' \rangle$ has at least 2 components.
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The Randomized approach
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The Contract Operation

**Function** \( \text{CONTRACT-EDGE}(G, e) \)

1. \{We will contract edge \( e \) in \( G \).\}
2. Let \( u \) and \( v \) denote the end-points of \( e \).
3. Identify \( u \) and \( v \) into a single vertex.
4. Remove all self-loops.
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**Note**

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Randomized Min-cut
The Randomized Algorithm

**Function** `RANDOMIZED_MIN-CUT(G = <V, E>)`

1. **while** there are more than 2 vertices in `G` **do**
2.   Pick an edge `e` uniformly and at random from the edge set `E`
3.   CONTRACT-EDGE(`G`, `e`)
4. **end while**
5. Let `a` and `b` denote the last two vertices that remain.
6. Output the edges between `a` and `b` as the min-cut of `G`. 
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Analysis (contd.)

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Coupon Collector problems
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(iii) Suppose you are given a $K$ coupons (usually less than $N$). What is the expected number of distinct coupon types in these $K$ coupons?
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Analyzing one coupon collection problem

We focus on the following question: What is the expected number of coupons to be collected, to ensure that each coupon type has been collected? Let $X$ denote the total number of coupons to be drawn, before we have at least one coupon of each of the $N$ distinct types. Let $X_i$ denote the number of coupons that need to be drawn, after $(i - 1)$ distinct coupon types have been collected, in order to draw a coupon of a type that has not been collected. If $p_i$ denotes the probability of drawing a new coupon after $(i - 1)$ coupons have been collected, then clearly, $p_i = \left(1 - \frac{i - 1}{N}\right) = \frac{N - i + 1}{N}$.

Now $X = \sum_{i=1}^{N} X_i$ and hence $E[X] = \sum_{i=1}^{N} E[X_i]$. Observe that each $X_i$ is a geometric random variable with parameter $p_i$. Therefore,

$$E[X] = \sum_{i=1}^{N} E[X_i]$$
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Now $X = \sum_{i=1}^{N} X_i$ and hence $E[X] = \sum_{i=1}^{N} E[X_i]$. Observe that each $X_i$ is a geometric random variable with parameter $p_i$. Therefore,

$$E[X] = \sum_{i=1}^{N} E[X_i]$$

$$= \sum_{i=1}^{N} \frac{1}{p_i}$$
Analysis (contd.)

Analyzing one coupon collection problem (contd.)
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\[
= \sum_{i=1}^{N} \frac{N}{N - i + 1}
\]
Analyzing one coupon collection problem (contd.)

\[
\begin{align*}
&= \sum_{i=1}^{N} \frac{N}{N - i + 1} \\
&= N \cdot \sum_{i=1}^{N} \frac{1}{i}
\end{align*}
\]
Analyzing one coupon collection problem (contd.)

\[
= \sum_{i=1}^{N} \frac{N}{N - i + 1}
\]

\[
= N \cdot \sum_{i=1}^{N} \frac{1}{i}
\]

\[
= N \cdot H_N
\]
Analyzing one coupon collection problem (contd.)

\[
\begin{align*}
= & \sum_{i=1}^{N} \frac{N}{N - i + 1} \\
= & N \cdot \sum_{i=1}^{N} \frac{1}{i} \\
= & N \cdot H_N \\
= & N \cdot \ln N + \Theta(N),
\end{align*}
\]
Analyzing one coupon collection problem (contd.)

\[
\begin{align*}
&= \sum_{i=1}^{N} \frac{N}{N - i + 1} \\
&= N \cdot \sum_{i=1}^{N} \frac{1}{i} \\
&= N \cdot H_N \\
&= N \cdot \ln N + \Theta(N), \text{ since } H_N = \ln N + \Theta(1)
\end{align*}
\]