Balls and Bins (Preliminaries)

K. Subramani\textsuperscript{1}

\textsuperscript{1}Lane Department of Computer Science and Electrical Engineering
West Virginia University

28 February, 2012
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1. The Birthday Paradox

2. Balls into Bins
Outline

1. The Birthday Paradox
2. Balls into Bins
3. The Poisson Distribution
   - Some important lemmas
   - Connection to Binomial Distribution
   - Connection to Balls and Bins
Overview
Main issues

We will study the experiment of throwing \( m \) balls into \( n \) bins,
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The Birthday Paradox

Balls into Bins

The Poisson Distribution
The Birthday Paradox

Experiment
The Birthday Paradox

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Model Assumptions

(i) Each year has exactly 365 days.
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(i) Each year has exactly 365 days.

(ii) Each person is equally likely to be born on any day.
The Birthday Paradox

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Model Assumptions

(i) Each year has exactly 365 days.

(ii) Each person is equally likely to be born on any day.

(iii) No twins or triplets or multiple people sharing the same birthday, from a pre-experiment perspective.
Analysis

We count configurations in which two people do not share a birthday.
Direct Counting

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- Total number of birthday configurations for the 30 people is $365^{30}$.
- In how many ways can you choose 30 distinct days from 365 days? $C(365, 30)$.
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- Total number of birthday configurations for the 30 people is $365^{30}$.
- In how many ways can you choose 30 distinct days from 365 days? $\binom{365}{30}$.
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- The probability that no two people share the same birthday is:
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$$q = \frac{C(365, 30) \cdot 30!}{365^{30}}$$
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- The required probability is therefore: $(1 - q)$. 
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We can compute the above probability by considering one person at a time. Consider an arbitrary order of the 30 people. Observe that the probability that the second person has a birthday that is distinct from the first person is: \(1 - \frac{1}{365}\).

Using the intersection lemma, we know that the probability that the \(k^{th}\) person has a birthday that is distinct from the first \((k - 1)\) birthdays, assuming that the first \((k - 1)\) people have distinct birthdays is:
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It follows that the probability that all 30 people have distinct birthdays is:

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q = \prod_{i=1}^{29} \left( 1 - \frac{i}{365} \right)
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The required probability is therefore \( 1 - q \). Detailed calculations show \( q \approx 0.2987 \), i.e., there is a better than 70% chance that two people share a birthday, when 30 people are in a room.
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The required probability is therefore \( 1 - q \). Detailed calculations show \( q \approx 0.2987 \), i.e., there is a better than 70% chance that two people share a birthday, when 30 people are in a room. Likewise, only 23 people need to be in the room, before the probability that two people share a birthday is more than \( \frac{1}{2} \).
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Assuming that there are $m$ people and $n$ possible birthdays.
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Assuming that there are $m$ people and $n$ possible birthdays. Recall that $1 - \frac{k}{n} \approx e^{-\frac{k}{n}}$, when $k \ll n$. The probability that all $m$ people have distinct birthdays is:

$$\prod_{j=1}^{m-1} \left(1 - \frac{j}{n}\right) \approx \prod_{j=1}^{m-1} e^{-\frac{j}{n}}$$

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= e^{-\frac{-m(m-1)}{2n}}
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Hence, the value for \( m \), at which the probability that all \( m \) people have distinct birthdays is \( \frac{1}{2} \) is
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Hence, the value for $m$, at which the probability that all $m$ people have distinct birthdays is $\frac{1}{2}$ is $m = \sqrt{2 \cdot n \cdot \ln 2}$. 
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Hence, the value for \( m \), at which the probability that all \( m \) people have distinct birthdays is \( \frac{1}{2} \) is \( m = \sqrt{2 \cdot n \cdot \ln 2} \). Check what you get, when \( n = 365! \).
Intuitive bounds
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If $k \leq \sqrt{n}$, this probability is less than $\frac{1}{2}$. 
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Subramani

Balls into Bins
Intuitive bounds (contd.)
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$$\left(1 - \frac{1}{\sqrt{n}}\right)^{\sqrt{n}} < \frac{1}{e}$$

$$< \frac{1}{2}$$

Hence, once there are $2 \cdot \sqrt{n}$ people, the probability is at most $\frac{1}{e}$, that the birthdays will be distinct.
The basic model

Measure of Interest

Consider the problem of throwing $m$ balls into $n$ bins, uniformly and at random.
The basic model

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Consider the problem of throwing $m$ balls into $n$ bins, uniformly and at random. The maximum load is defined as the maximum number of balls in any bin.
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## The basic model

### Measure of Interest

Consider the problem of throwing $m$ balls into $n$ bins, uniformly and at random. The maximum load is defined as the maximum number of balls in any bin. We will attempt to bound this quantity.

### Lemma

*When $m$ balls are thrown independently and uniformly at random into $n$ bins, the probability that the maximum load is more than $3 \cdot \frac{\ln n}{\ln \ln n}$ is at most $\frac{1}{n}$ for $n$ sufficiently large.*
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Note

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\frac{k^k}{k!} < \sum_{i=0}^{\infty} \frac{k^i}{i!}
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$$\frac{k^k}{k!} < \sum_{i=0}^{\infty} \frac{k^i}{i!} = e^k$$
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When $m$ balls are thrown independently and uniformly at random into $n$ bins, the probability that the maximum load is more than $3 \cdot \frac{\ln n}{\ln \ln n}$ is at most $\frac{1}{n}$ for $n$ sufficiently large.

Note

$$\frac{k^k}{k!} < \sum_{i=0}^{\infty} \frac{k^i}{i!} = e^k \Rightarrow k! > \left(\frac{k}{e}\right)^k$$
Balls and Bins (contd.)

Proof.
Balls and Bins (contd.)

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Bucket Sort
Bucket Sort

Main ideas
## Bucket Sort

### Main ideas

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The Poisson Distribution
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Definition

A discrete Poisson random variable $X$ with parameter $\mu > 0$ is given by the following probability distribution on $j = 0, 1, 2, \ldots$
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(i) Show that the definition leads to proper probability distribution.
(ii) What is $\mathbb{E}[X]$, when $X$ is a Poisson random variable?
The Birthday Paradox

Balls into Bins

The Poisson Distribution

Some important lemmas

Connection to Binomial Distribution

Connection to Balls and Bins
The sum lemma
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Let $X$ and $Y$ denote two Poisson random variables with means $\mu_1$ and $\mu_2$ respectively. Observe that,

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Moment Generating Function
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*The moment generating function of a Poisson random variable with parameter $\mu$ is $M_X(t) = e^{\mu \cdot (e^t - 1)}$.***
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For any $t$,

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\]

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For any \( t \),

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Theorem

Let $X$ be a Poisson random variable with parameter $\mu$. 
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- If $x < \mu$, then
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Plug in the mgf of the Poisson distribution to get,
Proof of Chernoff bounds

For any $t > 0$ and $x > \mu$, 

$$P(X \geq x) = P(e^{t \cdot X} \geq e^{t \cdot x})$$

$$\leq \frac{E[e^{t \cdot X}]}{e^{t \cdot x}}$$

Plug in the mgf of the Poisson distribution to get,

$$P(X \geq x) \leq e^{\mu \cdot (e^{t} - 1 - x \cdot t)}$$
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The complementary bound can be derived in similar fashion.
Outline

1. The Birthday Paradox
2. Balls into Bins
3. The Poisson Distribution
   - Some important lemmas
   - Connection to Binomial Distribution
   - Connection to Balls and Bins
Limit of the Binomial Distribution
Theorem

Let $X_n$ denote a binomial random variable with parameters $n$ and $p$, where $p$ is a function of $n$ and \( \lim_{n \to \infty} n \cdot p = \lambda \) is a constant that is independent of $n$. 
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Note
**Theorem**

Let \( X_n \) denote a binomial random variable with parameters \( n \) and \( p \), where \( p \) is a function of \( n \) and \( \lim_{n \to \infty} n \cdot p = \lambda \) is a constant that is independent of \( n \). Then, for any fixed \( k \),

\[
\lim_{n \to \infty} P(X_n = k) = \frac{e^{\lambda} \cdot \lambda^k}{k!}
\]

**Note**

- If \( |x| \leq 1 \),
Limit of the Binomial Distribution

**Theorem**

Let $X_n$ denote a binomial random variable with parameters $n$ and $p$, where $p$ is a function of $n$ and $\lim_{n \to \infty} n \cdot p = \lambda$ is a constant that is independent of $n$. Then, for any fixed $k$,

$$\lim_{n \to \infty} P(X_n = k) = \frac{e^\lambda \cdot \lambda^k}{k!}$$

**Note**

- If $|x| \leq 1$, $e^x \cdot (1 - x^2) \leq (1 + x) \leq e^x$. 
Theorem

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\lim_{n \to \infty} P(X_n = k) = \frac{e^\lambda \cdot \lambda^k}{k!}
$$

Note

- If $|x| \leq 1$, $e^x \cdot (1 - x^2) \leq (1 + x) \leq e^x$.
- $(1 - p)^k \geq (1 - p \cdot k)$, for $k \geq 0$. 

Limit of the Binomial Distribution
Proof
Proof.

\[ P(X_n = k) = \]
Proof

\[ P(X_n = k) = \binom{n}{k} \cdot p^k \cdot (1 - p)^{n-k} \]
Proof

\[ P(X_n = k) = C(n, k) \cdot p^k \cdot (1 - p)^{n-k} \leq \]
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\[ P(X_n = k) = C(n, k) \cdot p^k \cdot (1 - p)^{n-k} \]

\[ \leq \frac{n^k}{k!} \cdot p^k \cdot \frac{(1 - p)^n}{(1 - p)^k} \]
Proof.

\[ P(X_n = k) = \binom{n}{k} p^k (1 - p)^{n-k} \]

\[ \leq \frac{n^k}{k!} p^k (1 - p)^n \frac{1}{(1 - p)^k} \]
Proof.

\[ P(X_n = k) = C(n, k) \cdot p^k \cdot (1 - p)^{n-k} \]

\[ \leq \frac{n^k}{k!} \cdot p^k \cdot \frac{(1 - p)^n}{(1 - p)^k} \]

\[ \leq \frac{(n \cdot p)^k}{k!} \cdot e^{-p \cdot n} \cdot \frac{1}{1 - p \cdot k} \]
Proof

\[ P(X_n = k) = C(n, k) \cdot p^k \cdot (1 - p)^{n-k} \]

\[ \leq \frac{n^k}{k!} \cdot p^k \cdot \frac{(1 - p)^n}{(1 - p)^k} \]

\[ \leq \frac{(n \cdot p)^k}{k!} \cdot \frac{e^{-p\cdot n}}{1 - p \cdot k} \]

\[ = \frac{e^{-p\cdot n} \cdot (n \cdot p)^k}{k!} \cdot \frac{1}{1 - p \cdot k} \]
Proof.

\[ P(X_n = k) = C(n, k) \cdot p^k \cdot (1 - p)^{n-k} \]

\[ \leq \frac{n^k}{k!} \cdot \frac{p^k}{(1-p)^k} \cdot \frac{(1-p)^n}{(1-p)^k} \]

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Working similarly, we can show that,
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\[ \leq \frac{n^k}{k!} \cdot p^k \cdot \frac{(1-p)^n}{(1-p)^k} \]

\[ \leq \frac{(np)^k}{k!} \cdot \frac{e^{-p \cdot n}}{1 - p \cdot k} \]

\[ = \frac{e^{-p \cdot n} \cdot (np)^k}{k!} \cdot \frac{1}{1 - p \cdot k} \]

Working similarly, we can show that,

\[ P(X_n = k) \geq \]
Proof

\[ P(X_n = k) = \frac{n^k}{k!} \cdot \frac{p^k}{(1-p)^k} \cdot (1-p)^{n-k} \]

Working similarly, we can show that,

\[ P(X_n = k) \geq \frac{(n-k+1)^k}{k!} \cdot p^k \cdot (1-p)^n \]
Proof.

\[ P(X_n = k) = \binom{n}{k} \cdot p^k \cdot (1 - p)^{n-k} \]

\[ \leq \frac{n^k}{k!} \cdot p^k \cdot \frac{(1 - p)^n}{(1 - p)^k} \]

\[ \leq \frac{(n \cdot p)^k}{k!} \cdot \frac{e^{-p \cdot n}}{1 - p \cdot k} \]

\[ = \frac{e^{-p \cdot n} \cdot (n \cdot p)^k}{k!} \cdot \frac{1}{1 - p \cdot k} \]

Working similarly, we can show that,

\[ P(X_n = k) \geq \frac{(n - k + 1)^k}{k!} \cdot p^k \cdot (1 - p)^n \]

\[ \geq \]
Proof.

\[ P(X_n = k) = C(n, k) \cdot p^k \cdot (1 - p)^{n-k} \]

\[ \leq \frac{n^k}{k!} \cdot p^k \cdot \frac{(1 - p)^n}{(1 - p)^k} \]

\[ \leq \frac{(n \cdot p)^k}{k!} \cdot \frac{e^{-p \cdot n}}{1 - p \cdot k} \]

\[ = \frac{e^{-p \cdot n} \cdot (n \cdot p)^k}{k!} \cdot \frac{1}{1 - p \cdot k} \]

Working similarly, we can show that,

\[ P(X_n = k) \geq \frac{(n - k + 1)^k}{k!} \cdot p^k \cdot (1 - p)^n \]

\[ \geq \frac{e^{-p \cdot n} \cdot ((n - k + 1) \cdot p)^k}{k!} \cdot (1 - p^2 \cdot n) \]
Proof (contd.)
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Proof.
Proof.

Combining the above two inequalities gives us,

\[
e^{-p \cdot n} \cdot \frac{((n - k + 1) \cdot p)^k}{k!} \cdot (1 - p^2 \cdot n) \leq
\]

Proof (contd.)
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Combining the above two inequalities gives us,

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Proof. Combining the above two inequalities gives us,

\[
e^{-p\cdot n} \cdot (\frac{(n-k+1) \cdot p}{k!})^k \cdot (1 - p^2 \cdot n) \leq P(X_n = k) \leq \frac{e^{-p\cdot n} \cdot (n \cdot p)^k}{k!} \cdot \frac{1}{1 - p \cdot k}
\]

As \( n \) tends to \( \infty \), both the lower limit and the upper limit converge to \( \frac{e^{-\lambda \cdot \lambda^k}}{k!} \).
Outline

1. The Birthday Paradox

2. Balls into Bins

3. The Poisson Distribution
   - Some important lemmas
   - Connection to Binomial Distribution
   - Connection to Balls and Bins
Balls and Bins revisited
### Number of balls in a bin

<table>
<thead>
<tr>
<th>Bin</th>
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Balls and Bins revisited

Number of balls in a bin

What is the probability that a given bin is empty?
Balls and Bins revisited

Number of balls in a bin

What is the probability that a given bin is empty? \((1 - \frac{1}{n})^m\)
Balls and Bins revisited

Number of balls in a bin

What is the probability that a given bin is empty? \((1 - \frac{1}{n})^m \approx e^{-\frac{m}{n}}\).
Balls and Bins revisited

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What is the probability that a given bin is empty? \((1 - \frac{1}{n})^m \approx e^{-\frac{m}{n}}\). This probability is the same for all bins.
Balls and Bins revisited

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Let \(X_j\) be 1, if the \(j^{th}\) bin is empty and 0, otherwise.
Balls and Bins revisited

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What is the probability that a given bin is empty? \((1 - \frac{1}{n})^m \approx e^{-\frac{m}{n}}\). This probability is the same for all bins.

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Balls and Bins revisited

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\[
E[X] = \]

Subramani  Balls and Bins
Balls and Bins revisited

Number of balls in a bin

What is the probability that a given bin is empty? \((1 - \frac{1}{n})^m \approx e^{-\frac{m}{n}}\). This probability is the same for all bins.

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\[
E[X] = E\left[\sum_{i=1}^{n} X_i\right]
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The Birthday Paradox
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E[X] = E\left[\sum_{i=1}^{n} X_i\right]
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Balls and Bins revisited

**Number of balls in a bin**

What is the probability that a given bin is empty? \( (1 - \frac{1}{n})^m \approx e^{-\frac{m}{n}} \). This probability is the same for all bins.

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\[
E[X] = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E[X_i]
\]
Number of balls in a bin

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\[
E[X] = E[\sum_{i=1}^{n} X_i] \\
= \sum_{i=1}^{n} E[X_i] \\
\approx \sum_{i=1}^{n} \left(1 - \frac{1}{n}\right)^m
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\[
E[X] = E\left[\sum_{i=1}^{n} X_i\right]
\]
\[
= \sum_{i=1}^{n} E[X_i]
\]
\[
\approx n \cdot e^{-\frac{m}{n}}
\]
Balls and Bins revisited (contd.)
Generalization
Generalization

What is the probability that a given bin has $r$ balls?
Balls and Bins revisited (contd.)

Generalization

What is the probability that a given bin has \( r \) balls? \( C(m, r) \cdot \left( \frac{1}{n} \right)^r \cdot (1 - \frac{1}{n})^{m-r} \).
Generalization

What is the probability that a given bin has \( r \) balls? \( \binom{m}{r} \cdot \left(\frac{1}{n}\right)^r \cdot \left(1 - \frac{1}{n}\right)^{m-r} \).

This can be simplified to
What is the probability that a given bin has $r$ balls? $C(m, r) \cdot \left(\frac{1}{n}\right)^r \cdot (1 - \frac{1}{n})^{m-r}$.

This can be simplified to $p_r \approx e^{-\frac{m}{n}} \cdot \left(\frac{m}{n}\right)^r \cdot r!$. 
What is the probability that a given bin has $r$ balls? $C(m, r) \cdot \left(\frac{1}{n}\right)^r \cdot \left(1 - \frac{1}{n}\right)^{m-r}$.

This can be simplified to $p_r \approx \frac{e^{-\frac{m}{n}} \cdot \left(\frac{m}{n}\right)^r}{r!}$.

In other words, the number of balls in a specific bin is Poisson distributed with mean $\frac{m}{n}$. 

Generalization