Martingales

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17 April, 2012
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   - Introduction
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3. Wald's Equation
A sequence of random variables $Z_0, Z_1, \ldots$ is a martingale with respect to the sequence $X_0, X_1, \ldots$ if, for all $n \geq 0$, the following conditions hold:

(i) $Z_n$ is a function of $X_0, X_1, \ldots, X_n$;

(ii) $E[|Z_n|] < \infty$;

(iii) $E[Z_{n+1}|X_0, \ldots, X_n] = Z_n$.

A sequence of random variables $Z_0, Z_1, \ldots$ is called a martingale when it is a martingale with respect to itself. That is, $E[|Z_n|] < \infty$, and $E[Z_{n+1}|Z_0, \ldots, Z_n] = Z_n$.

Note 1: A Martingale can have a finite or a countably infinite number of elements.

Note 2: The indexing of the martingale sequence does not need to start at 0.
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Note

1. A Martingale can have a finite or a countably infinite number of elements.
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Consider a gambler who plays a sequence of fair games. Let $X_i$ be the amount the gambler wins on the $i$th game ($X_i$ is negative if the gambler loses), and let $Z_i$ be the gambler's total winnings at the end of the $i$th game. Because each game is fair, $E[X_i] = 0$. $E[Z_i + 1 | X_1, X_2, \ldots, X_i] = Z_i + E[X_{i+1}] = Z_i$.

Thus, $Z_1, Z_2, \ldots, Z_n$ is a martingale with respect to the sequence $X_1, X_2, \ldots, X_n$.

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Consider a gambler who plays a sequence of fair games. Let $X_i$ be the amount the gambler wins on the $i^{th}$ game ($X_i$ is negative if the gambler loses), and let $Z_i$ be the gambler’s total winnings at the end of the $i^{th}$ game.
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Doob Martingale

A Doob martingale refers to a martingale constructed using the following general approach. Let \( X_0, X_1, \ldots, X_n \) be a sequence of random variables, and let \( Y \) be a random variable with \( \mathbb{E}[|Y|] < \infty \). Then \( Z_i = \mathbb{E}[Y | X_0, \ldots, X_i] \), \( i = 0, 1, \ldots, n \), gives a martingale with respect to \( X_0, X_1, \ldots, X_n \), since

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**Note**

Observations
In most applications we start Doob martingale with $Z_0 = E[Y]$ which corresponds to $X_0$ being a trivial random variable that is independent of $Y$.

Consider that we want to estimate the value of $Y$, whose value is a function of the values of the random variables $X_1, \ldots, X_n$.

The sequence $Z_0, Z_1, \ldots, Z_n$ represents a sequence of refined estimates of the value of $Y$, gradually using more information on the values of the random variables $X_1, X_2, \ldots, X_n$. Then

1. $Z_0 = E[Y]$,
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Let $G$ be a random graph from $G_{n, p}$. We label all the $m = \binom{n}{2}$ possible edge slots in some arbitrary order, and let $X_j = \{1, \text{if there is an edge in the } j\text{th edge slot}, 0, \text{otherwise}\}$. Let $F(G)$ be the size of the largest independent set in $G$. The sequence $Z_0 = E[F(G)]$ and $Z_i = E[F(G) | X_1, \ldots, X_i]$ is a Doob martingale that represents the conditional expectations of $F(G)$ as we reveal whether each edge is in the graph, one edge at a time. This process of revealing edges gives a martingale called the edge exposure martingale.
Let $G$ be a random graph from $G_n, p$. We label all the $m = \binom{n}{2}$ possible edge slots in some arbitrary order, and let $X_j = \begin{cases} 1, & \text{if there is an edge in the } j\text{th edge slot,} \\ 0, & \text{otherwise.} \end{cases}$

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Doob martingale - Examples

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\end{cases}$$

Let $F(G)$ be the size of the largest independent set in $G$.

1. $Z_0 = \mathbb{E}[F(G)]$ and
2. $Z_i = \mathbb{E}[F(G) \mid X_1, \ldots, X_i]$

The sequence $Z_0, Z_1, \ldots, Z_m$ is a Doob martingale that represents the conditional expectations of $F(G)$ as we reveal whether each edge is in the graph, one edge at a time.

This process of revealing edges gives a martingale called the edge exposure martingale.
Instead of revealing edges one at a time, we could reveal the set of edges connected to a vertex, one vertex at a time. Consider the arbitrary numbering of vertices 1 through $n$, and let $G_i$ be the subgraph of $G$ induced by the first $i$ vertices. Then, setting $Z_0 = E[F(G)]$ and $Z_i = E[F(G) | G_1, \ldots, G_i]$ for $i = 1, \ldots, n$, gives a Doob martingale that is commonly called the vertex exposure martingale.
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Doob martingale - Examples

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Outline

1. Martingales
   - Definition
   - Doob Martingale

2. Stopping Times
   - Introduction
   - Martingale Stopping Theorem

3. Wald's Equation
Consider again the Gambler who participates in a sequence of fair gambling rounds, and $Z_i$ is the gambler's winnings after the $i$th game. If the player decides (before starting to play) to quit after exactly $k$ games, what are the gambler's expected winnings?

Lemma

If the sequence $Z_0, Z_1, \ldots, Z_n$ is a martingale with respect to $X_0, X_1, \ldots, X_n$, then $E[Z_n] = E[Z_0]$. 

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**Stopping Times**

**Introduction**

**Martingale Stopping Theorem**

**Wald's Equation**
Since $Z_0, Z_1, \ldots, Z_n$ is a martingale with respect to $X_0, X_1, \ldots, X_n$, it follows that

$$Z_i = E[Z_i + 1 | X_0, X_1, \ldots, X_i].$$

Taking the expectation on both sides and using the definition of conditional expectation, we have

$$E[Z_i] = E[E[Z_i + 1 | X_0, X_1, \ldots, X_i]].$$

Repeating this argument yields

$$E[Z_n] = E[Z_0].$$

Thus, if the number of games played is initially fixed then the expected gain from the sequence of games is zero.

Note The gambler could decide to keep playing until his winnings total at least a hundred dollars. The following notion proves quite powerful.
Proof

Since \( Z_0, Z_1, \ldots, Z_n \) is a martingale with respect to \( X_0, X_1, \ldots, X_n \), it follows that

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Stopping Times

Definition

A nonnegative, integer-valued random variable $T$ is a stopping time for the sequence $\{Z_n, n \geq 0\}$ if the event $T = n$ depends only on the value of the random variables $Z_0, Z_1, \ldots, Z_n$.

Which of the following are stopping times?

1. First time the gambler wins five games in a row - Stopping Time
2. First time the gambler has won at least a hundred dollars - Stopping Time
3. Last time the gambler wins five games in a row - Not a stopping time (needs knowledge of $Z_{n+1}, Z_{n+2}, \ldots$)

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Outline

1. Martingales
   - Definition
   - Doob Martingale

2. Stopping Times
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3. Wald's Equation
Theorem

If $Z_0, Z_1, \ldots$ is a martingale with respect to $X_1, X_2, \ldots$, and if $T$ is a stopping time for $X_1, X_2, \ldots$, then

$$E[Z_T] = E[Z_0]$$

whenever one of the following holds:

1. The $Z_i$ are bounded, so there is a constant $c$ such that, for all $i$, $|Z_i| \leq c$;
2. $T$ is bounded;
3. $E[T] < \infty$, and there is a constant $c$ such that $E[|Z_i + 1 - Z_i| |X_1, \ldots, X_i] < c$. 
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Consider a sequence of independent, fair gambling games. In each round, a player wins a dollar with probability 1/2 or loses a dollar with probability 1/2. Let $Z_0 = 0$, let $X_i$ be the amount won on the $i$th game, and let $Z_i$ be the total won by the player after $i$ games. Assume that the player quits the game when she either loses $l_1$ dollars or wins $l_2$ dollars. What is the probability that the player wins $l_2$ dollars before losing $l_1$ dollars?

Solution

Let $T$ be the first time the player has either won $l_2$ or lost $l_1$. Then $T$ is a stopping time for $X_1, X_2, \ldots$. The sequence $Z_0, Z_1, \ldots$ is a martingale, and since the values of $Z_i$ are clearly bounded we apply the martingale stopping theorem.

$$E[Z_T] = 0.$$ 

Let $q$ be the probability that the gambler quits playing after winning $l_2$ dollars. Then

$$E[Z_T] = l_2 \cdot q - l_1 \cdot (1 - q) = 0$$

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Let $T$ be the first time the player has either won $l_2$ or lost $l_1$. 

Let $q$ be the probability that the gambler quits playing after winning $l_2$ dollars. Then $E[Z_T] = l_2 \cdot q - l_1 \cdot (1 - q) = 0$ gives $q = \frac{l_1}{l_1 + l_2}$.
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$$E\left[\sum_{i=1}^{T} X_i\right] = E[T] \cdot E[X].$$

Proof

For $i \geq 1$, let $Z_i = i \sum_{j=1}^{i} (X_j - E[X])$. The sequence $Z_1, Z_2, \ldots$ is a martingale with respect to $X_1, X_2, \ldots$, and $E[Z_1] = 0$.

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Nitesh

Randomized Algorithms
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For \( i \geq 1 \), let \( Z_i = \sum_{j=1}^{i} (X_j - \mathbb{E}[X]) \).

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Proof

Hence on applying martingale stopping theorem

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Definition
Let $Z_0, Z_1, \ldots$ be a sequence of independent random variables. A non-negative, integer-valued random variable $T$ is a stopping time for the sequence if the event $T = n$ is independent of $Z_n + 1, Z_{n+2}, \ldots$

Example
Consider a gambling game in which a player first rolls one standard die. If the outcome of the roll is $X$ then she rolls $X$ new standard dice and her gain $Z$ is the sum of the outcome of the $X$ dice. What is the outcome of this game?

Solution
For $1 \leq i \leq X$, let $Y_i$ be the outcome of the $i$th die in the second round. Then

$$E[Z] = E[X \sum_{i=1}^X Y_i] = E[X] \cdot E[Y_i] = \left(\frac{7}{2}\right)^2 = \frac{49}{4}$$
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Las Vegas algorithms are the algorithms that always give the right answer but have variable running times. In a Las Vegas algorithm we often repeatedly perform some randomized subroutine that may or may not return the right answer. We then use some deterministic checking subroutine to determine whether or not the answer is correct; if correct the Las Vegas algorithm terminates with the correct answer, and otherwise the randomized subroutine is run again.

Application
Wald’s equation can be used in the analysis of Las Vegas algorithms. If $N$ is the number of trials until a correct answer is found and $X_i$ is the running time for the two subroutines on the $i$th trial, then according to Wald’s equation

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Wald’s equation can be used in the analysis of Las Vegas algorithms. If $N$ is the number of trials until a correct answer is found and $X_i$ is the running time for the two subroutines on the $i^{th}$ trial, then according to Wald’s equation

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\mathbb{E} \left[ \sum_{i=1}^{N} X_i \right] = \mathbb{E}[N] \cdot \mathbb{E}[X].
$$
Consider a set of $n$ servers communicating through a shared channel. Time is divided into discrete slots. At each time slot, any server that needs to send a packet can transmit it through the channel. If exactly one packet is sent at a time, the transmission is successfully completed. If more than one packet is sent, then none are successful. Packets are stored in the server’s buffer until they are successfully transmitted. At each time slot, if the server’s buffer is not empty then with probability $1/n$ it attempts to send the first packet in its buffer.

What is the expected number of time slots used until each server successfully sends at least one packet?

Solution

Let $N$ be the number of packets successfully sent until each server has successfully sent at least one packet. Let $t_i$ be the time slot in which the $i$th successfully transmitted packet is sent, starting from time $t_0 = 0$, and let $r_i = t_i - t_{i-1}$ and let $T$ be the number of time slots until each server successfully sends at least one packet, then $T = \sum_{i=1}^{N} r_i$. 
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The probability that a packet is successfully sent in a given time slot is 
\[ p = \left( \frac{1}{n} \right)^{n-1} \times \left( \frac{1}{n} \right) \times \left( 1 - \frac{1}{n} \right) \approx e^{-1} \]

The \( r_i \) each have a geometric distribution with parameter \( p \), so 
\[ E[r] = \frac{1}{p} \approx e. \]

Given that a packet was successfully sent at a given time slot, the sender of that packet is uniformly distributed among the \( n \) servers, independent of previous steps.

From the Coupon collector's problem, we deduce that 
\[ E[N] = n \cdot \log(n) + O(n). \]
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