Random Variables - Expectation

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4. Expectation of a function of a random variable
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Random Variables

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Example
Let $X$ denote the random variable that is defined as the sum of two fair dice.
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Let $X$ denote the random variable that is defined as the sum of two fair dice. What are the values that $X$ can take?
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$$P\{X = 1\} =$$
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$$P\{X = 1\} = 0$$
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Example

Let $X$ denote the random variable that is defined as the sum of two fair dice. What are the values that $X$ can take?

\[
P\{X = 1\} = 0 \\
P\{X = 2\} = \frac{1}{36}
\]
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\[
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&\quad \vdots \\
P\{X = 12\} &= \frac{1}{36}
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Definition

A random variable that can take on only a countable number of possible values is said to be *discrete*.
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Definition

A random variable that can take on only a countable number of possible values is said to be *discrete*. For a discrete random variable $X$, the probability mass function (pmf) $p(a)$ is defined as:

\[p(a) = P\{X = a\} \]
The Bernoulli Random Variable

Main idea

Consider an experiment which has exactly two outcomes;
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\[
p(1) = P\{X = 1\} = p \\
p(0) = P\{X = 0\} = 1 - p
\]

where $0 \leq p \leq 1$ is the probability that the experiment results in a success.
The Binomial Random Variable

Motivation

Consider an experiment which consists of \( n \) independent Bernoulli trials, with the probability of success in each trial being \( p \). If \( X \) is the random variable that counts the number of successes in the \( n \) trials, then \( X \) is said to be a Binomial Random Variable.
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Consider an experiment which consists of $n$ independent Bernoulli trials, with the probability of success in each trial being $p$. If $X$ is the random variable that counts the number of successes in the $n$ trials, then $X$ is said to be a Binomial Random Variable. The probability mass function of $X$ is given by:
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$$p(i) = P\{X = i\} = C(n, i) \cdot p^i \cdot (1 - p)^{n-i}, \quad i = 0, 1, 2, \ldots n$$
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Consider the experiment of tossing four fair coins.
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Consider the experiment of tossing four fair coins. What is the probability that you will get two heads and two tails?
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Example

Consider the experiment of tossing four fair coins. What is the probability that you will get two heads and two tails?
Example (contd.)

Solution

Let the event of heads turning up denote a “success.”
Example (contd.)

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Let the event of heads turning up denote a “success.” Accordingly, we are interested in the probability of getting exactly two successes in four Bernoulli trials.
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\[ p(2) = \binom{4}{2} \cdot \left(\frac{1}{2}\right)^2 \]
Solution

Let the event of heads turning up denote a “success.” Accordingly, we are interested in the probability of getting exactly two successes in four Bernoulli trials. As discussed above,

\[ p(2) = C(4, 2) \cdot \left( \frac{1}{2} \right)^2 \cdot \left( 1 - \frac{1}{2} \right)^2 \]
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Let the event of heads turning up denote a “success.” Accordingly, we are interested in the probability of getting exactly two successes in four Bernoulli trials. As discussed above,

\[ p(2) = \binom{4}{2} \cdot \left( \frac{1}{2} \right)^2 \cdot \left( 1 - \frac{1}{2} \right)^2 \]

\[ = \frac{3}{8} \]
The Geometric Random Variable

Motivation

Suppose that independent Bernoulli trials, each with probability $p$ of success are performed until a success occurs.
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The Geometric Random Variable

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$$p(i) = P\{X = i\} = (1 - p)^{i-1} \cdot p, \quad i = 1, 2, \ldots$$
Definition

Let $X$ denote a discrete random variable with probability mass function $p(x)$. 
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Note

\( E[X] \) is the weighted average of the possible values that \( X \) can assume, each value being weighted by the probability that \( X \) assumes that value.
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**Example**

Let $X$ denote the random variable that records the outcome of tossing a fair die.
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Expectation of a Bernoulli Random Variable

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Let $X$ denote a Bernoulli Random Variable with $p$ denoting the probability of success.
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**Solution:**

$$E[X] = 1 \cdot p + 0 \cdot (1 - p)$$
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$$= p$$
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Let $X$ denote a Binomial Random Variable with parameters $n$ and $p$. 
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$$E[X] = \sum_{i=0}^{n} i \cdot p(i), \text{ by definition}$$

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$$= n \cdot p \sum_{i=1}^{n} \frac{(n-1)!}{(i-1)!(n-i)!} \cdot p^{i-1} \cdot (1-p)^{n-i}$$
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= \sum_{i=0}^{n} i \cdot C(n, i) \cdot p^i \cdot (1 - p)^{n-i}
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\[
= n \cdot p \sum_{i=1}^{n} \frac{(n-1)!}{(i-1)!(n-i)!} \cdot p^{i-1} \cdot (1 - p)^{n-i}
\]

\( \square \)
Example

Substituting $k = i - 1$, we get,

$$E[X] = n \cdot p \sum_{k=0}^{n-1} \frac{(n-1)!}{k! \cdot (n-k-1)!} \cdot p^k \cdot (1-p)^{n-k-1}$$
Expectation of a Binomial Random Variable (contd.)

Example

Substituting \( k = i - 1 \), we get,

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E[X] = n \cdot p \sum_{k=0}^{n-1} \frac{(n-1)!}{k! \cdot (n-k-1)!} \cdot p^k \cdot (1-p)^{n-k-1}
\]

\[
= n \cdot p \sum_{k=0}^{n-1} \frac{(n-1)!}{k! \cdot ((n-1) - k)!} \cdot p^k \cdot (1-p)^{(n-1)-k}
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$$= n \cdot p \sum_{k=0}^{n-1} C(n - 1, k) \cdot p^k \cdot (1 - p)^{(n-1)-k}$$
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$$= n \cdot p \sum_{k=0}^{n-1} C(n-1, k) \cdot p^k \cdot (1-p)^{(n-1)-k}$$

$$= n \cdot p \cdot [p + (1-p)]^{n-1}, \text{ Binomial theorem}$$
Expectation of a Binomial Random Variable (contd.)

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= n \cdot p \cdot 1
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Substituting $k = i - 1$, we get,

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**Solution:**

$$E[X] = \sum_{i=1}^{\infty} i \cdot p(i), \text{ by definition}$$
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$$= \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1} \cdot p$$

$$= \sum_{i=1}^{\infty} i \cdot q^{i-1} \cdot p, \text{ where } q = 1 - p$$
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Solution:

$$E[X] = \sum_{i=1}^{\infty} i \cdot p(i), \text{ by definition}$$

$$= \sum_{i=1}^{\infty} i \cdot (1 - p)^{i-1} \cdot p$$

$$= \sum_{i=1}^{\infty} i \cdot q^{i-1} \cdot p, \text{ where } q = 1 - p$$

$$= p \cdot \sum_{i=1}^{\infty} i \cdot q^{i-1}$$
Recap
Random Variables
Expectation

Expectation of a function of a random variable
Linearity of Expectation
Conditional Expectation

Expectation of a Geometric Random Variable

Example

Let $X$ denote a Geometric Random Variable with parameters $n$ and $p$. What is $E[X]$?

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Expectation of a Geometric Random Variable (contd.)

Example

Solution:

\[ E[X] = p \cdot \frac{d}{dq} \left( \sum_{i=1}^{\infty} q^i \right) \]
Expectation of a Geometric Random Variable (contd.)

Example

Solution:

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Expectation of a Geometric Random Variable (contd.)

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Expectation of a Geometric Random Variable (contd.)

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Example

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Expectation of a function of a random variable
Linearity of Expectation
Conditional Expectation
Example

Consider the following game:
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Consider the following variation to the above game: The die is tossed till a 6 turns up. For each toss that does not turn up 6, A loses one dollar. If the toss turns up 6, A gets 6 dollars. How much money can A expect to make from this game?
Recap
Random Variables
Expectation
Expectation of a function of a random variable
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Exercises

Example
Consider the following game: A fair die is tossed. If the die turns up 6, person A wins one dollar. Otherwise, he loses a dollar. How much money can A expect to make from this game?

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Consider the following variation to the above game: The die is tossed till a 6 turns up. For each toss that does not turn up 6, A loses one dollar. If the toss turns up 6, A gets 6 dollars. How much money can A expect to make from this game?

Example

Consider yet another variation to the initial game: The die is tossed ten times. For each toss that turns up an even number, A gets 5 dollars. For tosses turning up an odd number, A loses 4 dollars. How much money can A expect to make from this game?
Expectation of a function of a random variable

Note

*Often times, we are interested in a function of the random variable, rather than the random variable itself.*
Recap
Random Variables
Expectation

Expectation of a function of a random variable
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Expectation of a function of a random variable

Note

Often times, we are interested in a function of the random variable, rather than the random variable itself. For instance, in the coin-tossing experiment, we could be interested in the square of the number of successes.
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Example

Let $X$ be a random variable, with the following pmf:

$$p(0) = 0.3, \ p(1) = 0.5, \ p(2) = 0.2$$
Expectation of a function of a random variable

**Note**

Often times, we are interested in a function of the random variable, rather than the random variable itself. For instance, in the coin-tossing experiment, we could be interested in the square of the number of successes. The question of interest then is how to determine the expectation of a function of a random variable, given that we only know the distribution of the random variable.

**Example**

Let $X$ be a random variable, with the following pmf:

\[ p(0) = 0.3, \ p(1) = 0.5, \ p(2) = 0.2 \]

Compute $E[X^2]$. 

\[ E[X^2] = 0 \cdot 0.3 + 1 \cdot 0.5 + 4 \cdot 0.2 = 0.7 \]
Expectation of functions of random variables (contd.)

**Solution**

Let $Y = X^2$. 
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Let $Y = X^2$. Observe that $Y$ is also a random variable. What are the values that $Y$ can take? 0, 1 and 4. Let us compute the pmf of $Y$. Note that,

$$P\{Y = 0\} = P\{X^2 = 0\} = P\{X = 0\} = 0.3$$
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Let $Y = X^2$. Observe that $Y$ is also a random variable. What are the values that $Y$ can take? 0, 1 and 4. Let us compute the pmf of $Y$. Note that,

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Solution

Let $Y = X^2$. Observe that $Y$ is also a random variable. What are the values that $Y$ can take? 0, 1 and 4. Let us compute the pmf of $Y$. Note that,

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Accordingly,

$$E[Y] = E[X^2] = 0 \cdot 0.3 + 1 \cdot 0.5 + 4 \cdot 0.2 = 1.3$$
Expectation of functions - The Direct Approach
Expectation of functions - The Direct Approach

**Theorem**

If $X$ is a random variable with pmf $p()$, and $g()$ is any real-valued function, then,

$$E[g(X)] = \sum_x g(x) \cdot p(x)$$
Theorem

If $X$ is a random variable with pmf $p()$, and $g()$ is any real-valued function, then,

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Proof.

Exercise.
Expectation of functions - The Direct Approach

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Applying the above theorem to the previous problem,
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Applying the above theorem to the previous problem,

$$E[X^2] =$$
Expectation of functions - The Direct Approach

**Theorem**

*If $X$ is a random variable with pmf $p()$, and $g()$ is any real-valued function, then,*

$$E[g(X)] = \sum_x g(x) \cdot p(x)$$

**Proof.**

Exercise.

**Note**

*Applying the above theorem to the previous problem,*

$$E[X^2] = 0^2 \cdot 0.3 + 1^2 \cdot 0.5 + 2^2 \cdot 0.2 = 1.3$$
Linearity of Expectation

Proposition
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Let $X_1, X_2, \ldots, X_n$ denote $n$ random variables, defined over some probability space.
Linearity of Expectation

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Let $X_1, X_2, \ldots, X_n$ denote $n$ random variables, defined over some probability space. Let $a_1, a_2, \ldots, a_n$ denote $n$ constants. Then,
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Let \( X_1, X_2, \ldots, X_n \) denote \( n \) random variables, defined over some probability space. Let \( a_1, a_2, \ldots, a_n \) denote \( n \) constants. Then,

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E\left[\sum_{i=1}^{n} a_i \cdot X_i\right] = \sum_{i=1}^{n} a_i \cdot E[X_i]
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Linearity of Expectation

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Let $X_1, X_2, \ldots, X_n$ denote $n$ random variables, defined over some probability space. Let $a_1, a_2, \ldots, a_n$ denote $n$ constants. Then,

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**Lemma**

Linearity of Expectation

Proposition

Let $X_1, X_2, \ldots, X_n$ denote $n$ random variables, defined over some probability space. Let $a_1, a_2, \ldots, a_n$ denote $n$ constants. Then,

$$E\left[\sum_{i=1}^{n} a_i \cdot X_i\right] = \sum_{i=1}^{n} a_i \cdot E[X_i]$$

Lemma


Lemma

Let $X$ denote a random variable and let $c$ denote a constant. Then, $E[c \cdot X] = c \cdot E[X]$. 

Subramani

Probability Theory
Note

Note that linearity of expectation holds even when the random variables are not independent.
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**Theorem**

If $X$ and $Y$ are independent random variables, then $E[X \cdot Y] = E[X] \cdot E[Y]$. 
### Linearity of Expectation (contd.)

**Note**

*Note that linearity of expectation holds even when the random variables are not independent.*

**Theorem**

*If $X$ and $Y$ are independent random variables, then $E[X \cdot Y] = E[X] \cdot E[Y]$.***

**Example**

What is the expected value of the sum of the upturned faces, when two fair dice are tossed?
Another Application

Example

Compute the expected value of the Binomial random variable.
Another Application

Example

Compute the expected value of the Binomial random variable.

Solution

Define

\[ X_i = 1, \]
Another Application

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\[ X_i = 1, \text{ if the } i^{th} \text{ trial is a success} \]
Another Application

Example

Compute the expected value of the Binomial random variable.

Solution

Define

\[ X_i = \begin{cases} 
1, & \text{if the } i^{\text{th}} \text{ trial is a success} \\
0, & \text{otherwise} 
\end{cases} \]
Another Application

Example

Compute the expected value of the Binomial random variable.

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\[ X_i = \begin{cases} 1, & \text{if the } i^{\text{th}} \text{ trial is a success} \\ 0, & \text{otherwise} \end{cases} \]

Accordingly, the Binomial random variable (say \( X \)) can be expressed as:

\[ X = X_1 + X_2 + \ldots + X_n \]
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Jensen’s inequality
Observation

What is the relation between $E[X^2]$ and $(E[X])^2$?
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Definition

Convex function
Jensen’s inequality

**Observation**

*What is the relation between \( E[X^2] \) and \( (E[X])^2 \)?*

**Definition**

Convex function - A function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is said to be convex,
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**Jensen’s inequality**

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a convex function, and $X$ is a random variable, then

$$f(E[X]) \leq E[f(X)]$$
Conditional Expectation
Conditional Expectation

**Definition**

Let $X$ and $Y$ denote two random variables. The conditional expectation of $X$, given that $Y = y$, is defined as follows:

$$E[X \mid Y = y] = \sum_x x \cdot Pr(X = x \mid Y = y).$$
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**Example**

Let $X_1$ and $X_2$ denote the random variables monitoring the upturned faces of two tossed dice and let $X = X_1 + X_2$. 
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Let $X_1$ and $X_2$ denote the random variables monitoring the upturned faces of two tossed dice and let $X = X_1 + X_2$. What is $E[X \mid X_1 = 2]$ and $E[X_1 \mid X = 5]$?
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**Theorem**

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**Conditional Expectation**

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**Theorem**
Let $X$ and $Y$ denote two random variables. Then,

$$E[X] = \sum_y Pr(Y = y) \cdot E[X \mid Y = y]$$
Conditional Expectation (contd.)

Proof.
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Conditional Expectation (contd.)

Proof.

Observe that,

$$\sum_y \Pr(Y = y) \cdot E[X \mid Y = y] =$$
Proof.

Observe that,

\[ \sum_y \Pr(Y = y) \cdot E[X \mid Y = y] = \sum_y \Pr(Y = y) \cdot \sum_x x \cdot \Pr(X = x \mid Y = y) \]
Proof.

Observe that,

\[ \sum_y \Pr(Y = y) \cdot E[X \mid Y = y] = \sum_y \Pr(Y = y) \cdot \sum_x x \cdot \Pr(X = x \mid Y = y) \]

\[ = \sum_x \sum_y x \cdot \Pr(X = x \mid Y = y) \cdot \Pr(Y = y) \]
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Conditional Expectation (contd.)

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\[
= \sum_x x \cdot \Pr(X = x)
\]

\[
= E[X]
\]
Conditional Expectation (contd.)
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Definition

The expression $E[X \mid Y]$ is a random variable and takes on the values $E[X \mid Y = y]$, when $Y = y$. 
Conditional Expectation (contd.)

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Theorem
Let $X$ and $Y$ denote any two random variables.
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Theorem

Let \( X \) and \( Y \) denote any two random variables. Then,

\[
E[X] = E[E[X \mid Y]]
\]