Markov Chains and Stationary Distributions

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Outline

1 Stationary Distributions
   - Fundamental Theorem of Markov Chains
   - Computing Stationary Distributions
   - Example: A Simple Queue
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   - Fundamental Theorem of Markov Chains
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2 Random Walks on Undirected Graphs
   - Application: An s-t Connectivity Algorithm
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   - Fundamental Theorem of Markov Chains
   - Computing Stationary Distributions
   - Example: A Simple Queue

2. Random Walks on Undirected Graphs
   - Application: An s-t Connectivity Algorithm

3. Parrondo’s Paradox
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1 Stationary Distributions
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2 Random Walks on Undirected Graphs
   - Application: An s-t Connectivity Algorithm

3 Parrondo's Paradox
Recall

Let \( \mathbf{P} \) be a one-step probability matrix of a Markov chain such that

\[
\mathbf{P} = \begin{pmatrix}
P_{0,0} & P_{0,1} & \cdots & P_{0,j} & \cdots \\
P_{1,0} & P_{1,1} & \cdots & P_{1,j} & \cdots \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
P_{i,0} & P_{i,1} & \cdots & P_{i,j} & \cdots \\
\vdots & \vdots & \ddots & \vdots & \ddots 
\end{pmatrix}.
\]
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Let $\bar{\mathbf{p}}(t) = (p_0(t), p_1(t), p_2(t), \ldots)$ be the vector giving the probability distribution of the state of the chain at time $t$. 

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Then, $\bar{\mathbf{p}}(t) = \bar{\mathbf{p}}(t-1) \cdot \mathbf{P}$. 

We are now interested in state probability distributions that do not change after a transition.
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Let $\bar{p}(t) = (p_0(t), p_1(t), p_2(t), \ldots)$ be the vector giving the probability distribution of the state of the chain at time $t$. Then,

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We are now interested in state probability distributions that do not change after a transition.
A \textbf{stationary distribution} (also called an equilibrium distribution) of a Markov chain is a probability distribution $\pi$ such that

$$\pi = \pi \cdot P.$$
**Definition**

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**Notes**

- If a chain reaches a stationary distribution, then it maintains that distribution for all future time.
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- If a chain reaches a stationary distribution, then it maintains that distribution for all future time.
- A stationary distribution represents a steady state (or an equilibrium) in the chain’s behavior.
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**Notes**

- If a chain reaches a stationary distribution, then it maintains that distribution for all future time.
- A stationary distribution represents a steady state (or an equilibrium) in the chain’s behavior.
- Stationary distributions play a key role in analyzing Markov chains.
Example

Suppose we have a Markov chain having state space \( S = \{0, 1, 2\} \) and transition matrix

\[
P = \begin{bmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\
\frac{1}{6} & \frac{1}{3} & \frac{1}{2}
\end{bmatrix}.
\]

The stationary distribution \( \pi \) of this Markov chain is \( \pi_0 = \frac{6}{25}, \pi_1 = \frac{10}{25}, \pi_2 = \frac{9}{25} \).

What does this mean?

Consider the total time spent once the chain reaches the stationary distribution.

\( \frac{6}{25} = 24\% \) of the time is spent in state 0.

\( \frac{10}{25} = 40\% \) of the time is spent in state 1.

\( \frac{9}{25} = 36\% \) of the time is spent in state 2.
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P = \begin{pmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{4}{6} & \frac{2}{3} & \frac{1}{2} \\
\frac{1}{6} & \frac{3}{2} & \frac{1}{1}
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We discuss first the case of finite chains and then extend the results to any discrete space chain.
Fundamental Theorem of Markov Chains

- We discuss first the case of finite chains and then extend the results to any discrete space chain.
- Without loss of generality, assume that the finite set of states of the Markov chain is \( \{0, 1, \ldots, n\} \).
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Theorem

Any finite, irreducible, and ergodic Markov chain has the following properties:
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Theorem

Any finite, irreducible, and ergodic Markov chain has the following properties:

1. The chain has a unique stationary distribution \( \vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_n) \).
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Any finite, irreducible, and ergodic Markov chain has the following properties:

1. The chain has a unique stationary distribution \( \vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_n) \).
2. For all \( j \) and \( i \), the limit \( \lim_{t \to \infty} P^t_{j,i} \) exists, and it is independent of \( j \).
Fundamental Theorem of Markov Chains

We discuss first the case of finite chains and then extend the results to any discrete space chain.

Without loss of generality, assume that the finite set of states of the Markov chain is \( \{0, 1, \ldots, n\} \).

Theorem

Any finite, irreducible, and ergodic Markov chain has the following properties:

1. The chain has a unique stationary distribution \( \tilde{\pi} = (\pi_0, \pi_1, \ldots, \pi_n) \).
2. For all \( j \) and \( i \), the limit \( \lim_{t \to \infty} P^t_{j,i} \) exists, and it is independent of \( j \).
3. \( \pi_i = \lim_{t \to \infty} P^t_{j,i} = \frac{1}{h_{j,i}} \), where \( h_{j,i} \) is the expected time to return to state \( i \) when starting at state \( i \).
The stationary distribution \( \bar{\pi} \) has two interpretations:

1. \( \pi_i \) is the limiting probability that the Markov chain will be in state \( i \) infinitely far out in the future. This probability is independent of the initial state. If we run the chain long enough, the initial state of the chain is almost forgotten, and the probability of being in state \( i \) converges to \( \pi_i \).

2. \( \pi_i \) is the inverse of \( h_{i,i} \), where \( h_{i,i} = \sum_{t=1}^{\infty} t \cdot r_{t,i} \). If the average time to return to state \( i \) from \( i \) is \( h_{i,i} \), then we expect to be in state \( i \) for \( \frac{1}{h_{i,i}} \) of the time and thus, in the limit, we must have \( \pi_i = \frac{1}{h_{i,i}} \).
Intuition

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Intuition

The stationary distribution \( \pi \) has two interpretations:

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2. \( \pi_i \) is the inverse of \( h_{i,i} = \sum_{t=1}^{\infty} t \cdot r_{i,i}^t \).
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1. $\pi_i$ is the limiting probability that the Markov chain will be in state $i$ infinitely far out in the future.
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2. $\pi_i$ is the inverse of $h_{i,i} = \sum_{t=1}^{\infty} t \cdot r_{i,i}^t$.
   - If the average time to return to state $i$ from $i$ is $h_{i,i}$, then we expect to be in state $i$ for $\frac{1}{h_{i,i}}$ of the time and thus, in the limit, we must have $\pi_i = \frac{1}{h_{i,i}}$. 
Proof of Fundamental Theorem

Lemma

For any irreducible, ergodic Markov chain and for any state $i$, the limit $\lim_{t \to \infty} P_{i,i}^t$ exists and

$$\lim_{t \to \infty} P_{i,i}^t = \frac{1}{h_{i,i}}.$$
Proof of Fundamental Theorem

**Lemma**

For any irreducible, ergodic Markov chain and for any state \(i\), the limit \(\lim_{t \to \infty} P_{i,i}^t\) exists and

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**Intuition**

- Since the expected time between visits to \(i\) is \(h_{i,i}\), state \(i\) is visited \(1/h_{i,i}\) of the time.
Proof of Fundamental Theorem

Lemma

For any irreducible, ergodic Markov chain and for any state $i$, the limit \( \lim_{t \to \infty} P^t_{i,i} \) exists and

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\lim_{t \to \infty} P^t_{i,i} = \frac{1}{h_{i,i}}.
\]

Intuition

- Since the expected time between visits to $i$ is $h_{i,i}$, state $i$ is visited $1/h_{i,i}$ of the time.
- Thus, \( \lim_{t \to \infty} P^t_{i,i} \), which represents the probability a state chosen far in the future is at state $i$ when the chain starts at state $i$, must be $1/h_{i,i}$. 

Proof of Fundamental Theorem

Showing Limits Exist and are Independent of Starting State $j$

Using the fact that $\lim_{t \to \infty} P^t_{j,i}$ exists, for any $j$ and $i$,

$$
\lim_{t \to \infty} P^t_{j,i} = \lim_{t \to \infty} P^t_{i,i} = \frac{1}{h_{i,i}}.
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Proof of Fundamental Theorem

Showing Limits Exist and are Independent of Starting State $j$

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$$\lim_{t \to \infty} P_{j,i}^t = \lim_{t \to \infty} P_{i,i}^t = \frac{1}{h_{i,i}}.$$

Proof

Read pages 168-169 in the book!
Proof of Fundamental Theorem

Proving $\bar{\pi}$ Forms a Stationary Distribution

Let $\pi_i = \lim_{t \to \infty} P_{j,i}^t = \frac{1}{h_{i,i}}$. 

For every $t \geq 0$, we have $P_{j,i}^t \geq 0$ and thus $\pi_i \geq 0$. For any $t \geq 0$, $\sum_i P_{j,i}^t = 1$ and thus $\lim_{t \to \infty} \sum_i P_{j,i}^t = \sum_i \pi_i = 0$ $\lim_{t \to \infty} P_{j,i}^t = \sum_i \pi_i = \frac{1}{h_{i,i}}$. $\bar{\pi}$ is a proper distribution.
### Proof of Fundamental Theorem of Markov Chains

#### Proving $\pi$ Forms a Stationary Distribution

- Let $\pi_i = \lim_{t \to \infty} P_{j,i}^t = \frac{1}{h_{i,i}}$.

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Proof of Fundamental Theorem

Proving $\pi$ Forms a Stationary Distribution

- Let $\pi_i = \lim_{t \to \infty} P_{j,i}^t = \frac{1}{h_{i,i}}$.

- For every $t \geq 0$, we have $P_{i,i}^t \geq 0$ and thus $\pi_i \geq 0$.

- For any $t \geq 0$, $\sum_{i=0}^{n} P_{j,i}^t = 1$ and thus

$$\lim_{t \to \infty} \sum_{i=0}^{n} P_{j,i}^t = \sum_{i=0}^{n} \lim_{t \to \infty} P_{j,i}^t =$$
Proof of Fundamental Theorem

Proving $\overline{\pi}$ Forms a Stationary Distribution

- Let $\pi_i = \lim_{t \to \infty} P_{j,i}^t = \frac{1}{h_{i,j}}$.
- For every $t \geq 0$, we have $P_{i,i}^t \geq 0$ and thus $\pi_i \geq 0$.
- For any $t \geq 0$, $\sum_{i=0}^{n} P_{j,i}^t = 1$ and thus

$$\lim_{t \to \infty} \sum_{i=0}^{n} P_{j,i}^t = \sum_{i=0}^{n} \lim_{t \to \infty} P_{j,i}^t = \sum_{i=0}^{n} \pi_i =$$
Proof of Fundamental Theorem

Proving $\pi$ Forms a Stationary Distribution

- Let $\pi_i = \lim_{t \to \infty} P_{j,i}^t = \frac{1}{h_{i,i}}$.
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- For any $t \geq 0$, $\sum_{i=0}^{n} P_{j,i}^t = 1$ and thus

$$\lim_{t \to \infty} \sum_{i=0}^{n} P_{j,i}^t = \sum_{i=0}^{n} \lim_{t \to \infty} P_{j,i}^t = \sum_{i=0}^{n} \pi_i = 1$$
Proof of Fundamental Theorem

Proving $\vec{\pi}$ Forms a Stationary Distribution

- Let $\pi_i = \lim_{t \to \infty} P_{j,i}^t = \frac{1}{h_{i,i}}$.
- For every $t \geq 0$, we have $P_{i,i}^t \geq 0$ and thus $\pi_i \geq 0$.
- For any $t \geq 0$, $\sum_{i=0}^{n} P_{j,i}^t = 1$ and thus $\lim_{t \to \infty} \sum_{i=0}^{n} P_{j,i}^t = \sum_{i=0}^{n} \lim_{t \to \infty} P_{j,i}^t = \sum_{i=0}^{n} \pi_i = 1$
- $\vec{\pi}$ is a proper distribution.
Proof of Fundamental Theorem

Proving $\bar{\pi}$ Forms a Stationary Distribution

Now,

$$P^{t+1}_{j,i} = \sum_{k=0}^{n} P^t_{j,k} \cdot P_{k,i}.$$
Proof of Fundamental Theorem

Proving $\pi$ Forms a Stationary Distribution

- Now,

$$P^{t+1}_{j,i} = \sum_{k=0}^{n} P^t_{j,k} \cdot P_{k,i}.$$ 

- Letting $t \to \infty$, we have

$$P^{t+1}_{j,i} =$$
Proof of Fundamental Theorem

Proving $\pi$ Forms a Stationary Distribution

- Now,

$$P_{j,i}^{t+1} = \sum_{k=0}^{n} P_{j,k}^t \cdot P_{k,i}.$$  

- Letting $t \to \infty$, we have

$$P_{j,i}^{t+1} = \pi_j$$
Proof of Fundamental Theorem

Proving \( \pi \) Forms a Stationary Distribution

- Now,

\[
P_{j,i}^{t+1} = \sum_{k=0}^{n} P_{j,k}^t \cdot P_{k,i}.
\]

- Letting \( t \to \infty \), we have

\[
P_{j,i}^{t+1} = \pi_j
\]

\[
P_{j,k}^t =
\]
Proof of Fundamental Theorem

Proving \( \bar{\pi} \) Forms a Stationary Distribution

- Now,

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P_{j,i}^{t+1} = \sum_{k=0}^{n} P_{j,k}^t \cdot P_{k,i}.
\]

- Letting \( t \to \infty \), we have

\[
\begin{align*}
P_{j,i}^{t+1} &= \pi_j \\
P_{j,k}^t &= \pi_k
\end{align*}
\]
Proof of Fundamental Theorem

Proving $\overline{\pi}$ Forms a Stationary Distribution

- Now,
  
  $$P_{j,i}^{t+1} = \sum_{k=0}^{n} P_{j,k}^{t} \cdot P_{k,i}.$$  

- Letting $t \to \infty$, we have
  
  $$P_{j,i}^{t+1} = \overline{\pi}_j$$  
  $$P_{j,k}^{t} = \overline{\pi}_k$$  
  $$\overline{\pi}_i = \sum_{k=0}^{n} \overline{\pi}_k \cdot P_{k,i}.$$
Proof of Fundamental Theorem

Proving $\overline{\pi}$ Forms a Stationary Distribution

Now,

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Letting $t \to \infty$, we have

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$$\pi_i = \sum_{k=0}^{n} \pi_k \cdot P_{k,i}$$

Therefore, $\overline{\pi}$ is a stationary distribution.
Proof of Fundamental Theorem

Proving $\bar{\pi}$ Forms a Stationary Distribution

- Now,

$$P_{j,i}^{t+1} = \sum_{k=0}^{n} P_{j,k}^{t} \cdot P_{k,i}.$$ 

- Letting $t \to \infty$, we have

$$P_{j,i}^{t+1} = \pi_j$$
$$P_{j,k}^{t} = \pi_k$$
$$\pi_j = \sum_{k=0}^{n} \pi_k \cdot P_{k,i}$$

Therefore, $\bar{\pi}$ is a stationary distribution.
Proof of Fundamental Theorem

Proving the Stationary Distribution is Unique

Suppose there were another stationary distribution \( \vec{\phi} \).
Proof of Fundamental Theorem

Proving the Stationary Distribution is Unique

- Suppose there were another stationary distribution \( \overline{\phi} \).
- By the same argument, we would have

\[
\phi_i = \sum_{k=0}^{n} \phi_k \cdot P_{k,i}^t.
\]

Since \( \sum_{k=0}^{n} \phi_k = 1 \), it follows that \( \phi_i = \pi_i \) for all \( i \), or \( \overline{\phi} = \overline{\pi} \).
Proof of Fundamental Theorem

Proving the Stationary Distribution is Unique

- Suppose there were another stationary distribution $\bar{\phi}$.
- By the same argument, we would have

$$\phi_i = \sum_{k=0}^{n} \phi_k \cdot P_{k,i}.$$

- Taking the limit at $t \to \infty$ yields

$$\phi_i = \sum_{k=0}^{n} \phi_k.$$
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$$\phi_i = \sum_{k=0}^{n} \phi_k \pi = \pi_i \sum_{k=0}^{n} \phi_k.$$
Proof of Fundamental Theorem

Proving the Stationary Distribution is Unique

- Suppose there were another stationary distribution \( \bar{\phi} \).
- By the same argument, we would have
  \[
  \phi_i = \sum_{k=0}^{n} \phi_k \cdot P_{k,i}^t.
  \]
- Taking the limit at \( t \to \infty \) yields
  \[
  \phi_i = \sum_{k=0}^{n} \phi_k \pi_i = \pi_i \sum_{k=0}^{n} \phi_k.
  \]
- Since \( \sum_{k=0}^{n} \phi_k \) =
Proof of Fundamental Theorem

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- Suppose there were another stationary distribution $\bar{\phi}$.
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$$\phi_i = \sum_{k=0}^{n} \phi_k \pi_i = \cdot \pi_i \sum_{k=0}^{n} \phi_k.$$ 

- Since $\sum_{k=0}^{n} \phi_k = 1$, 

$$\phi_i = \pi_i \sum_{k=0}^{n} \phi_k.$$
Proving the Stationary Distribution is Unique

- Suppose there were another stationary distribution $\bar{\phi}$.
- By the same argument, we would have
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  \]
- Since $\sum_{k=0}^{n} \phi_k = 1$, it follows that $\phi_i = \pi_i$ for all $i$, or $\bar{\phi} = \bar{\pi}$. 
Outline

1 Stationary Distributions
   - Fundamental Theorem of Markov Chains
   - Computing Stationary Distributions
   - Example: A Simple Queue

2 Random Walks on Undirected Graphs
   - Application: An s-t Connectivity Algorithm

3 Parrondo’s Paradox
One way to compute the stationary distribution of a finite Markov chain is to solve the system of linear equations

$$\pi \cdot P = \pi.$$

If we are given a transition matrix $P$, we have the following system

$$\pi_0 = \pi_0 \cdot P_{0,0} + \pi_1 \cdot P_{1,0} + \cdots + \pi_i \cdot P_{i,0} + \cdots$$

$$\pi_1 = \pi_0 \cdot P_{0,1} + \pi_1 \cdot P_{1,1} + \cdots + \pi_i \cdot P_{i,1} + \cdots$$

...
One way to compute the stationary distribution of a finite Markov chain is to solve the system of linear equations

$$\bar{\pi} \cdot P = \bar{\pi}.$$ 

If we are given a transition matrix

$$P = \begin{bmatrix} P_{0,0} & P_{0,1} & \cdots & P_{0,j} & \cdots \\ P_{1,0} & P_{1,1} & \cdots & P_{1,j} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P_{i,0} & P_{i,1} & \cdots & P_{i,j} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \end{bmatrix},$$

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One way to compute the stationary distribution of a finite Markov chain is to solve the system of linear equations

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One way to compute the stationary distribution of a finite Markov chain is to solve the system of linear equations
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\[
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P_{0,0} & P_{0,1} & \cdots & P_{0,j} & \cdots \\
P_{1,0} & P_{1,1} & \cdots & P_{1,j} & \cdots \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
P_{i,0} & P_{i,1} & \cdots & P_{i,j} & \cdots \\
\vdots & \vdots & \ddots & \vdots & \ddots 
\end{bmatrix},
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we have the following system
\[
\begin{align*}
\pi_0 &= \pi_0 \cdot P_{0,0} + \pi_1 \cdot P_{1,0} + \cdots + \pi_i \cdot P_{i,0} + \cdots \\
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    P_{1,0} & P_{1,1} & \cdots & P_{1,j} & \cdots \\
    \vdots & \vdots & \ddots & \vdots & \ddots \\
    P_{i,0} & P_{i,1} & \cdots & P_{i,j} & \cdots \\
    \vdots & \vdots & \ddots & \vdots & \ddots 
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\pi_j &= \pi_0 \cdot P_{0,j} + \pi_1 \cdot P_{1,j} + \cdots + \pi_i \cdot P_{i,j} + \cdots
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\]
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    P_{1,0} & P_{1,1} & \cdots & P_{1,j} & \cdots \\
    \vdots & \vdots & \ddots & \vdots & \ddots \\
    P_{i,0} & P_{i,1} & \cdots & P_{i,j} & \cdots \\
    \vdots & \vdots & \ddots & \vdots & \ddots 
\end{bmatrix}, \]

we have the following system

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\begin{align*}
\pi_0 &= \pi_0 \cdot P_{0,0} + \pi_1 \cdot P_{1,0} + \cdots + \pi_i \cdot P_{i,0} + \cdots \\
\pi_1 &= \pi_0 \cdot P_{0,1} + \pi_1 \cdot P_{1,1} + \cdots + \pi_i \cdot P_{i,1} + \cdots \\
\pi_j &= \pi_0 \cdot P_{0,j} + \pi_1 \cdot P_{1,j} + \cdots + \pi_i \cdot P_{i,j} + \cdots \quad \text{Done?}
\end{align*}
\]
Computing Stationary Distributions

System of Linear Equations

- One way to compute the stationary distribution of a finite Markov chain is to solve the system of linear equations

\[ \bar{\pi} \cdot P = \bar{\pi}. \]

- If we are given a transition matrix

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\vdots & \vdots & \ddots & \vdots & \ddots \\
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\vdots & \vdots & \ddots & \vdots & \ddots
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\pi_j &= \pi_0 \cdot P_{0,j} + \pi_1 \cdot P_{1,j} + \cdots + \pi_i \cdot P_{i,j} + \cdots \\
1 &= \pi_0 + \pi_1 + \cdots + \pi_j + \cdots
\end{align*}
\]
Exercise

Three out of every four trucks on the road are followed by a car, while only one out of every five cars is followed by a truck.
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Solution

Imagine sitting on the side of the road watching vehicles go by.
Exercise

Three out of every four trucks on the road are followed by a car, while only one out of every five cars is followed by a truck. What fraction of vehicles on the road are trucks?

Solution

- Imagine sitting on the side of the road watching vehicles go by.
- If a truck goes by, the next vehicle will be a car with probability $\frac{3}{4}$ and will be a truck with probability $\frac{1}{4}$. 
Exercise

Three out of every four trucks on the road are followed by a car, while only one out of every five cars is followed by a truck. What fraction of vehicles on the road are trucks?

Solution

- Imagine sitting on the side of the road watching vehicles go by.
- If a truck goes by, the next vehicle will be a car with probability 3/4 and will be a truck with probability 1/4.
- If a car goes by, the next vehicle will be a car with probability 4/5 and will be a truck with probability 1/5.
Exercise

Three out of every four trucks on the road are followed by a car, while only one out of every five cars is followed by a truck. What fraction of vehicles on the road are trucks?

Solution

- Imagine sitting on the side of the road watching vehicles go by.
- If a truck goes by, the next vehicle will be a car with probability $\frac{3}{4}$ and will be a truck with probability $\frac{1}{4}$.
- If a car goes by, the next vehicle will be a car with probability $\frac{4}{5}$ and will be a truck with probability $\frac{1}{5}$.
- Let 0 be the state that the vehicle is a truck, and let 1 be the state that the vehicle is a car.
Our transition probability matrix is

\[
P = \begin{bmatrix}
1 & 4 \\
3 & 4 \\
1 & 5 \\
4 & 5
\end{bmatrix}
\]

Our system of equations is:

\[
\pi_0 = \frac{1}{4} \cdot \pi_0 + \frac{1}{5} \cdot \pi_1 \\
\pi_1 = \frac{3}{4} \cdot \pi_0 + \frac{4}{5} \cdot \pi_1 \\
1 = \pi_0 + \pi_1
\]

Solving the first equation gives us:

\[
\frac{3}{4} \cdot \pi_0 = \frac{1}{5} \cdot \pi_1 \\
\pi_0 = \frac{4}{15} \cdot \pi_1
\]
Solution (Contd.)

Our transition probability matrix is

\[
P = \begin{bmatrix}
1 & 3 \\
\frac{4}{4} & \frac{4}{4} \\
\frac{1}{5} & \frac{4}{5}
\end{bmatrix}.
\]
Solution (Contd.)

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P = \begin{bmatrix}
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Our system of equations is

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\pi_0 = \frac{1}{4} \pi_0 + \frac{1}{5} \pi_1 \\
\pi_1 = \frac{3}{4} \pi_0 + \frac{4}{5} \pi_1 \\
\pi_0 + \pi_1 = 1.
\]
Solution (Contd.)

Our transition probability matrix is

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Our system of equations is

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\pi_0 &= \frac{1}{4} \cdot \pi_0 + \frac{1}{5} \cdot \pi_1 \\
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\]

Solving the first equation gives us
Solution (Contd.)

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$$P = \begin{bmatrix} 1 & 3 \\ \frac{1}{4} & \frac{4}{4} \\ \frac{5}{5} & \frac{5}{5} \end{bmatrix}.$$  

Our system of equations is

$$\pi_0 = \frac{1}{4} \cdot \pi_0 + \frac{1}{5} \cdot \pi_1$$
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Solving the first equation gives us

$$\frac{3}{4} \cdot \pi_0 = \frac{1}{5} \cdot \pi_1.$$
Solution (Contd.)

Our transition probability matrix is

\[ P = \begin{bmatrix}
  1 & 3 \\
  4 & 4 \\
  5 & 5 
\end{bmatrix}. \]

Our system of equations is

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\begin{align*}
\pi_0 &= \frac{1}{4} \cdot \pi_0 + \frac{1}{5} \cdot \pi_1 \\
\pi_1 &= \frac{3}{4} \cdot \pi_0 + \frac{4}{5} \cdot \pi_1 \\
1 &= \pi_0 + \pi_1.
\end{align*}
\]

Solving the first equation gives us

\[
\frac{3}{4} \cdot \pi_0 = \frac{1}{5} \cdot \pi_1 \\
\pi_0 = \frac{1}{15} \cdot \pi_1.
\]
Plugging this into the constraint $\pi_0 + \pi_1 = 1$, we get
Solution (Contd.)

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Plugging this into the constraint $\pi_0 + \pi_1 = 1$, we get

$$\frac{4}{15} \cdot \pi_1 + \pi_1 = 1$$

$$\frac{19}{15} \cdot \pi_1 = 1$$

Therefore, $\pi_0 = \frac{4}{19}$.
Solution (Contd.)

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\frac{4}{15} \cdot \pi_1 + \pi_1 = 1
\]
\[
\frac{19}{15} \cdot \pi_1 = 1
\]
\[
\pi_1 = \frac{15}{19}.
\]

Therefore, $\pi_0 = \frac{4}{19}$.

Thus, as we sit by the road, $\frac{4}{19}$ of all vehicles passing by will be trucks.
Solution (Contd.)

Plugging this into the constraint $\pi_0 + \pi_1 = 1$, we get

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Plugging this into the constraint $\pi_0 + \pi_1 = 1$, we get

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\pi_1 = \frac{15}{19}.
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Thus, as we sit by the road, $\frac{4}{19}$ of all vehicles passing by will be trucks.
Solution (Contd.)

Plugging this into the constraint \( \pi_0 + \pi_1 = 1 \), we get

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\frac{4}{15} \cdot \pi_1 + \pi_1 = 1 \\
\frac{19}{15} \cdot \pi_1 = 1 \\
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\]

Therefore, \( \pi_0 = \frac{4}{19} \). Thus, as we sit by the road, \( \frac{4}{19} \) of all vehicles passing by will be trucks.
Another technique is to study the cut-sets of the Markov chain.
Another technique is to study the cut-sets of the Markov chain. For any state $i$ of the chain,

$$\sum_{j=0}^{n} \pi_j \cdot P_{j,i} = \pi_i = \pi_i \sum_{j=0}^{n} P_{i,j}$$

or

$$\sum_{j \neq i} \pi_j \cdot P_{j,i} = \sum_{j \neq i} \pi_i P_{i,j}.$$
Another technique is to study the cut-sets of the Markov chain. For any state $i$ of the chain,

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In the stationary distribution, the probability that a chain leaves a state equals the probability that it enters the state.
Cut Sets

- Another technique is to study the cut-sets of the Markov chain.
- For any state $i$ of the chain,

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- In the stationary distribution, the probability that a chain leaves a state equals the probability that it enters the state.
- This observation can be generalized to sets of states.
### Cut Sets

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### Theorem

Let $S$ be a set of states of a finite, irreducible, aperiodic Markov chain.
Computing Stationary Distributions

Cut Sets

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- For any state \( i \) of the chain,

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\sum_{j=0}^{n} \pi_j \cdot P_{j,i} = \pi_i = \pi_i \sum_{j=0}^{n} P_{i,j}
\]

or

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\sum_{j \neq i} \pi_j \cdot P_{j,i} = \sum_{j \neq i} \pi_i P_{i,j}.
\]

- In the stationary distribution, the probability that a chain leaves a state equals the probability that it enters the state.
- This observation can be generalized to sets of states.

Theorem

Let \( S \) be a set of states of a finite, irreducible, aperiodic Markov chain. In the stationary distribution, the probability that the chain leaves the set \( S \) equals the probability that it enters \( S \).
If $C$ is a cut-set in the graph representation of the chain, then in the stationary distribution, the probability of crossing the cut-set in one direction is equal to the probability of crossing the cut-set in the other direction.
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Example

![Diagram of a simple queue example](image-url)
Note

If $C$ is a cut-set in the graph representation of the chain, then in the stationary distribution, the probability of crossing the cut-set in one direction is equal to the probability of crossing the cut-set in the other direction.

Example

From state 0, we move to state 1 with probability $p$ and stay at state 0 with probability $1 - p$. From state 1, we move to state 0 with probability $q$ and stay at state 1 with probability $1 - q$. 
### Note

If $C$ is a cut-set in the graph representation of the chain, then in the stationary distribution, the probability of crossing the cut-set in one direction is equal to the probability of crossing the cut-set in the other direction.

### Example

From state 0, we move to state 1 with probability $p$ and stay at state 0 with probability $1 - p$.

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Note

If $C$ is a cut-set in the graph representation of the chain, then in the stationary distribution, the probability of crossing the cut-set in one direction is equal to the probability of crossing the cut-set in the other direction.

Example

- From state 0, we move to state 1 with probability $p$ and stay at state 0 with probability $1 - p$.
- From state 1, we move to state 0 with probability $q$ and stay at state 1 with probability $1 - q$.
- If $p$ and $q$ are small, state changes are rare.
Our transition matrix is

\[
P = \begin{bmatrix}
1 - p & p \\
q & 1 - q
\end{bmatrix}
\]

Solving \( \bar{\pi} \cdot P = \bar{\pi} \) corresponds to solving the system

\[
\pi_0 \cdot (1 - p) + \pi_1 \cdot q = \pi_0 \\
\pi_0 \cdot q + \pi_1 \cdot (1 - q) = \pi_1
\]

Our solution is \( \pi_0 = \frac{q}{p + q} \) and \( \pi_1 = \frac{p}{p + q} \).
Our transition matrix is

\[
P = \begin{bmatrix}
1 - p & p \\
q & 1 - q
\end{bmatrix}
\]

Using System of Equations

Our solution is \( \pi_0 = \frac{q}{p + q} \) and \( \pi_1 = \frac{p}{p + q} \).
Computing Stationary Distributions

Using System of Equations

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Computing Stationary Distributions

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Solving \( \vec{\pi} \cdot P = \vec{\pi} \) corresponds to solving the system

\[
\begin{align*}
\pi_0 \cdot (1 - p) + \pi_1 \cdot q &= \pi_0 \\
\pi_0 \cdot q + \pi_1 \cdot (1 - q) &= \pi_1 \\
\pi_0 + \pi_1 &= 1
\end{align*}
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Computing Stationary Distributions

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\pi_0 = \frac{q}{p + q} \quad \text{and} \quad \pi_1 = \frac{p}{p + q}.
\]
### Computing Stationary Distributions

**Using System of Equations**

Our transition matrix is

\[
P = \begin{bmatrix} 1 - p & p \\ q & 1 - q \end{bmatrix}
\]

Solving \( \vec{\pi} \cdot P = \vec{\pi} \) corresponds to solving the system

\[
\begin{align*}
\pi_0 \cdot (1 - p) + \pi_1 \cdot q &= \pi_0 \\
\pi_0 \cdot q + \pi_1 \cdot (1 - q) &= \pi_1 \\
\pi_0 + \pi_1 &= 1
\end{align*}
\]

Our solution is

\[
\pi_0 = \frac{q}{p + q} \quad \text{and} \quad \pi_1 = \frac{p}{p + q}.
\]

**Using Cut-Set Formulation**

The probability of leaving state 0 must equal the probability of entering state 0, or
## Computing Stationary Distributions

### Using System of Equations

Our transition matrix is

\[
P = \begin{bmatrix}
1 - p & p \\
q & 1 - q
\end{bmatrix}
\]

Solving \( \vec{\pi} \cdot P = \vec{\pi} \) corresponds to solving the system

\[
\begin{align*}
\pi_0 \cdot (1 - p) + \pi_1 \cdot q &= \pi_0 \\
\pi_0 \cdot q + \pi_1 \cdot (1 - q) &= \pi_1 \\
\pi_0 + \pi_1 &= 1
\end{align*}
\]

Our solution is

\[
\pi_0 = \frac{q}{p + q} \quad \text{and} \quad \pi_1 = \frac{p}{p + q}.
\]

### Using Cut-Set Formulation

The probability of leaving state 0 must equal the probability of entering state 0, or

\[
\pi_0 \cdot p = \]

Computing Stationary Distributions

Using System of Equations

Our transition matrix is

$$P = \begin{bmatrix} 1 - p & p \\ q & 1 - q \end{bmatrix}$$

Solving $\bar{\pi} \cdot P = \bar{\pi}$ corresponds to solving the system

$$\begin{align*}
\pi_0 \cdot (1 - p) + \pi_1 \cdot q &= \pi_0 \\
\pi_0 \cdot q + \pi_1 \cdot (1 - q) &= \pi_1 \\
\pi_0 + \pi_1 &= 1
\end{align*}$$

Our solution is

$$\pi_0 = \frac{q}{p + q} \quad \text{and} \quad \pi_1 = \frac{p}{p + q}.$$ 

Using Cut-Set Formulation

The probability of leaving state 0 must equal the probability of entering state 0, or

$$\pi_0 \cdot p = \pi_1 \cdot q.$$
Computing Stationary Distributions

Using System of Equations

Our transition matrix is

$$P = \begin{bmatrix} 1 - p & p \\ q & 1 - q \end{bmatrix}$$

Solving $\vec{\pi} \cdot P = \vec{\pi}$ corresponds to solving the system

$$\begin{align*}
\pi_0 \cdot (1 - p) + \pi_1 \cdot q &= \pi_0 \\
\pi_0 \cdot q + \pi_1 \cdot (1 - q) &= \pi_1 \\
\pi_0 + \pi_1 &= 1
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Using $\pi_0 + \pi_1 = 1$ yields $\pi_0 = \frac{q}{p + q}$ and $\pi_1 = \frac{p}{p + q}$.
Theorem

Consider a finite, irreducible, and ergodic Markov chain with transition matrix $P$. 

Proof

Consider the $j$th entry of $\bar{\pi} \cdot P$.

Using the assumption of the theorem, we find that it equals $\sum_{i=0}^{n} \pi_i \cdot P_{i,j} = \sum_{i=0}^{n} \pi_j \cdot P_{j,i} = \pi_j$.

Thus $\bar{\pi}$ satisfies $\bar{\pi} = \bar{\pi} \cdot P$.

Since $\sum_{i=0}^{n} \pi_i = 1$, it follows from the Fundamental Theorem that $\bar{\pi}$ must be the unique stationary distribution of the Markov chain.

Definition

Chains that satisfy the condition $\pi_i \cdot P_{i,j} = \pi_j \cdot P_{j,i}$ are called time reversible.

Williamson

Markov Chains and Stationary Distributions
Theorem

Consider a finite, irreducible, and ergodic Markov chain with transition matrix $\mathbf{P}$. If there are nonnegative numbers $\bar{\pi} = (\pi_0, \ldots, \pi_n)$ such that $\sum_{i=0}^{n} \pi_i = 1$ and if, for any pair of states $i, j$,
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**Stationary Distributions**

**Random Walks on Undirected Graphs**

**Parrondo’s Paradox**

**Fundamental Theorem of Markov Chains**

**Computing Stationary Distributions**

**Example: A Simple Queue**

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### Computing Stationary Distributions

**Theorem**

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Chains that satisfy the condition $\pi_i \cdot P_{i,j} = \pi_j \cdot P_{j,i}$ are called **time reversible**.
Convergence of Markov Chains with Countably Infinite State Spaces

Theorem

Any irreducible aperiodic Markov chain belongs to one of the following two categories:
Convergence of Markov Chains with Countably Infinite State Spaces

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Any irreducible aperiodic Markov chain belongs to one of the following two categories:

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Proof

Same as the proof of the Fundamental Theorem.
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Same as the proof of the Fundamental Theorem.
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### Convergence of Markov Chains with Countably Infinite State Spaces

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Outline

1. **Stationary Distributions**
   - Fundamental Theorem of Markov Chains
   - Computing Stationary Distributions
   - Example: A Simple Queue

2. **Random Walks on Undirected Graphs**
   - Application: An $s$-$t$ Connectivity Algorithm

3. **Parrondo’s Paradox**
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Queue Example

Queues

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If $X_t$ is the number of customers in the queue at time $t$, then all the $X_t$ yield a finite-state Markov chain.
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- $P_{i,i+1} = \lambda$ if $i < n$
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- $P_{i,i} = \begin{cases} 1 - \lambda & \text{if } i = 0 \\ 1 - \lambda - \mu & \text{if } 1 \leq i \leq n - 1 \\ 1 - \mu & \text{if } i = n \end{cases}$
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Markov Chain

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Queue Example

Solution

A solution of the system is

\[ \pi_i = \frac{(\lambda/\mu)^i}{\sum_{i=0}^{\infty} (\lambda/\mu)^i} = \left( \frac{\lambda}{\mu} \right)^i \cdot \left( 1 - \frac{\lambda}{\mu} \right). \]

This generalizes the solution to the case where there is an upper bound \( n \) on the number of the customers in the system

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Queue Example

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All of the $\pi_i$ are greater than 0 if and only if $\lambda < \mu \Rightarrow$ the rate at which customers arrive is lower than the rate customers are served.
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- All of the \( \pi_i \) are greater than 0 if and only if \( \lambda < \mu \Rightarrow \) the rate at which customers arrive is lower than the rate customers are served.

- If \( \lambda > \mu \), the rate at which customers arrive is higher than the rate they depart.
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- This means there is no stationary distribution, and the queue length will become arbitrarily long.
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- If \( \lambda > \mu \), the rate at which customers arrive is higher than the rate they depart.
- This means there is no stationary distribution, and the queue length will become arbitrarily long.
- In this case, each state in the Markov chain is transient.
- If \( \lambda = \mu \), there is still no stationary distribution, but the states are null recurrent.
Outline

1. Stationary Distributions
   - Fundamental Theorem of Markov Chains
   - Computing Stationary Distributions
   - Example: A Simple Queue

2. Random Walks on Undirected Graphs
   - Application: An s-t Connectivity Algorithm

3. Parrondo’s Paradox
A random walk on an undirected graph is a special type of Markov chain that is often used in analyzing algorithms.
Random Walks

Introduction

- A random walk on an undirected graph is a special type of Markov chain that is often used in analyzing algorithms.
- Let $G = (V, E)$ be a finite, undirected, and connected graph.
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**Definition**

A **random walk** on $G$ is a Markov chain defined by the sequence of moves of a particle between vertices of $G$. In this process, the place of the particle at a given time step is the state of the system.
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A **random walk** on $G$ is a Markov chain defined by the sequence of moves of a particle between vertices of $G$. In this process, the place of the particle at a given time step is the state of the system. If the particle is at vertex $i$ and if $i$ has $d(i)$ outgoing edges, then the probability that the particle follows the edge $(i, j)$ and moves to a neighbor $j$ is $1/d(i)$. 
Random Walks

Example

Undirected Graph $G$

$\begin{array}{c}
\text{a} \\
\text{c} \\
\text{b}
\end{array}$
Random Walks

Example

Undirected Graph $G$

Markov Chain

Undirected Graph $G$
A random walk on an undirected graph $G$ is aperiodic if and only if $G$ is not bipartite.
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Proof:
- A graph is bipartite if and only if it does not have cycles with an odd number of edges.
Lemma

A random walk on an undirected graph $G$ is aperiodic if and only if $G$ is not bipartite.

Proof

- A graph is bipartite if and only if it does not have cycles with an odd number of edges.
- In an undirected graph, there is always a path of length 2 from a vertex to itself.
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A random walk on an undirected graph $G$ is aperiodic if and only if $G$ is not bipartite.

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- A graph is bipartite if and only if it does not have cycles with an odd number of edges.
- In an undirected graph, there is always a path of length 2 from a vertex to itself.
- If the graph is bipartite, then the random walk is periodic with period $d = 2$. 

Williamson
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- In an undirected graph, there is always a path of length 2 from a vertex to itself.
- If the graph is bipartite, then the random walk is periodic with period $d = 2$.
- If the graph is not bipartite, what can you say about the cycles?
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Criterion for Aperiodicity

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- It follows that the Markov chain is aperiodic.
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- If the graph is not bipartite, what can you say about the cycles? The graph has an odd cycle.
- By traversing that cycle we have an odd-length path from any vertex to itself.
- It follows that the Markov chain is aperiodic.

**Note**
A random walk on a finite, undirected, connected, and non-bipartite graph $G$ satisfies the conditions of our Fundamental Theorem which means the random walk converges to a stationary distribution.
Theorem

A random walk on \( G \) converges to a stationary distribution \( \bar{\pi} \), where

\[
\pi_v = \frac{d(v)}{2 \cdot |E|}.
\]
Theorem

A random walk on $G$ converges to a stationary distribution $\bar{\pi}$, where

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Proof

Since $\sum_{v \in V} d(v) = 2 \cdot |E|$, it follows that

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and $\bar{\pi}$ is a proper distribution over $v \in V$. 

Williamson

Markov Chains and Stationary Distributions
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\[\text{Williamson}\] 

\[\text{Markov Chains and Stationary Distributions}\]
Stationary Distribution

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  and $\pi$ is a proper distribution over $v \in V$.
- Let $P$ be the transition probability matrix of the Markov chain.
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- Let $P$ be the transition probability matrix of the Markov chain.
- Let $N(v)$ represent the neighbors of $v$. 

Williamson  
Markov Chains and Stationary Distributions
Proof

The relation $\bar{\pi} = \bar{\pi} \cdot P$ is equivalent to

$$\pi_v = \sum_{u \in N(v)} \frac{d(u)}{2 \cdot |E|} \cdot \frac{1}{d(u)}$$
Stationary Distribution

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Note

Recall that $h_{v,u}$ denotes the expected number of steps to reach $u$ from $v$. 
Stationary Distribution

Proof

The relation $\tilde{\pi} = \pi \cdot P$ is equivalent to

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Note

Recall that $h_{v,u}$ denotes the expected number of steps to reach $u$ from $v$.

Corollary

For any vertex $u$ in $G$,

$$
h_{u,u} = \frac{2 \cdot |E|}{d(u)}.
$$
Lemma

If \((u, v) \in E\), then \(h_{v,u} < 2 \cdot |E|\).
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Let \(N(u)\) be the set of neighbors of vertex \(u\) in \(G\).
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Proof

Let \(N(u)\) be the set of neighbors of vertex \(u\) in \(G\). We compute \(h_{u,u}\) in two different ways:
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\frac{2 \cdot |E|}{d(u)} = h_{u,u} = \frac{1}{d(u)} \cdot \sum_{w \in N(u)} (1 + h_{w,u}).
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**Definition**
The **cover time** of a graph \(G = (V, E)\) is the maximum expected time to visit all of the vertices in the graph by a random walk starting from \(v\), for all vertices \(v \in V\).
Lemma

The cover time of $G = (V, E)$ is bounded above by $4 \cdot |V| \cdot |E|$. 
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Proof

Choose a spanning tree of $G$; any subset of of the edges that gives an acyclic graph connecting all of the vertices in $G$. 
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- Choose a spanning tree of $G$; any subset of of the edges that gives an acyclic graph connecting all of the vertices in $G$.
- There exists a cyclic tour on this spanning tree in which every edge is traversed once in each direction.
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- Choose a spanning tree of $G$; any subset of of the edges that gives an acyclic graph connecting all of the vertices in $G$.
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- Let $v_0, v_1, \ldots, v_{2|V|-2} = v_0$ be the sequence of vertices in the tour, starting from $v_0$. 
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- Clearly, the expected time to go through the vertices in the tour is an upper bound on the cover time.
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- Hence the cover time is bounded above by

$$\sum_{i=0}^{2|V|-3} h_{v_i, v_{i+1}}$$
Cover Time

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$$2 \cdot |V| - 3 \sum_{i=0}^{2|V| - 3} h_{v_i, v_{i+1}} < (2|V| - 2) \cdot (2 \cdot |E|)$$
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Hence the cover time is bounded above by

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\sum_{i=0}^{2|V|-3} h_{v_i, v_{i+1}} < (2|V| - 2) \cdot (2 \cdot |E|) < 4 \cdot |V| \cdot |E|.
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Outline

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Connectivity Algorithm

Problem Description

- Suppose we are given an undirected graph $G = (V, E)$ and two vertices $s$ and $t$ in $G$. 
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Let $n = |V|$ and $m = |E|$. 

Perform a random walk on $G$ for enough steps so that a path from $s$ to $t$ is likely to be found. We use the cover time result to bound the number of steps that the random walk has to run.

Assume $G$ is non-bipartite.
Problem Description

- Suppose we are given an undirected graph $G = (V, E)$ and two vertices $s$ and $t$ in $G$.
- Let $n = |V|$ and $m = |E|$.
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This can be done in linear time using breadth-first search or depth-first search.
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- These approaches require $\Omega(n)$ space.
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Randomized Algorithm

- Our randomized algorithm works with only $O(\log n)$ bits of memory.
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- We use the cover time result to bound the number of steps that the random walk has to run.
- Assume \( G \) is non-bipartite.
**Function** $s$-$t$ CONNECTIVITY($s$, $t$)

1. Start a random walk from $s$.
2. if (walk reaches $t$ within $4 \cdot n^3$ steps) then
3.      return (“There is a path”)
4. else
5.      return (“There is no path”)
6. end if

**Algorithm 3.1:** $s$-$t$ Connectivity Algorithm
Theorem

The $s$-$t$ connectivity algorithm returns the correct answer with probability $1/2$, and it only errs by returning that there is no path from $s$ to $t$ when there is such a path.
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The $s$-$t$ connectivity algorithm returns the correct answer with probability $1/2$, and it only errs by returning that there is no path from $s$ to $t$ when there is such a path.

Proof

- If there is no path, does the algorithm return an incorrect answer?
Theorem

The \( s-t \) connectivity algorithm returns the correct answer with probability 1/2, and it only errs by returning that there is no path from \( s \) to \( t \) when there is such a path.

Proof

- If there is no path, does the algorithm return an incorrect answer? No!
Connectivity Algorithm

Theorem
The \( s-t \) connectivity algorithm returns the correct answer with probability 1/2, and it only errs by returning that there is no path from \( s \) to \( t \) when there is such a path.

Proof
- If there is no path, does the algorithm return an incorrect answer? No!
- If there is a path, the algorithm errs if it does not find a path within \( 4 \cdot n^3 \) steps of the walk.
Connectivity Algorithm

Theorem
The \( s-t \) connectivity algorithm returns the correct answer with probability \( 1/2 \), and it only errs by returning that there is no path from \( s \) to \( t \) when there is such a path.

Proof
- If there is no path, does the algorithm return an incorrect answer? No!
- If there is a path, the algorithm errs if it does not find a path within \( 4 \cdot n^3 \) steps of the walk.
- The expected time to reach \( t \) from \( s \) is bounded from above by the cover time of their shared component, which is at most
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4 \cdot n \cdot m
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4 \cdot n \cdot m < 4 \cdot n \cdot \left( \frac{n \cdot (n-1)}{2} \right)
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- The probability that a walk takes more than $4 \cdot n^3$ steps to reach $s$ from $t$ is at most

$$P(X > 4 \cdot n^3) \leq$$
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\[ P(X > 4 \cdot n^3) \leq \frac{2 \cdot n^3}{4 \cdot n^3}. \]
Theorem

The s-t connectivity algorithm returns the correct answer with probability 1/2, and it only errs by returning that there is no path from s to t when there is such a path.

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$$4 \cdot n \cdot m < 4 \cdot n \cdot \left( \frac{n \cdot (n - 1)}{2} \right) < 2 \cdot n^3.$$

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$$P(X > 4 \cdot n^3) \leq \frac{2 \cdot n^3}{4 \cdot n^3} = \frac{1}{2}.$$
The algorithm must keep track of its current position, which takes $O(\log n)$ bits, as well as the number of steps taken in the random walk, which also takes only $O(\log n)$ bits.
Connectivity Algorithm

Notes

- The algorithm must keep track of its current position, which takes $O(\log n)$ bits, as well as the number of steps taken in the random walk, which also takes only $O(\log n)$ bits.
- As long as there is some mechanism for choosing a random neighbor from each vertex, that is all the memory required.
Outline

1. Stationary Distributions
   - Fundamental Theorem of Markov Chains
   - Computing Stationary Distributions
   - Example: A Simple Queue

2. Random Walks on Undirected Graphs
   - Application: An s-t Connectivity Algorithm

3. Parrondo’s Paradox
Parrondo’s Paradox

Main Idea

Given two games, each with a higher probability of losing than winning, it is possible to construct a winning strategy by playing the games alternately.
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How does this apply to Markov Chains?

- Let $A$ and $B$ be the two games.
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- Let $A$ and $B$ be the two games.
- Use Markov Chains to show that both $A$ and $B$ are losing games by analyzing absorbing states or using stationary distributions.
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- Let $A$ and $B$ be the two games.
- Use Markov Chains to show that both $A$ and $B$ are losing games by analyzing absorbing states or using stationary distributions.
- Combine games $A$ and $B$ into a new game $C$, where you alternate between games $A$ and $B$ with a provided probability.
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- Use Markov Chains to show that both $A$ and $B$ are losing games by analyzing absorbing states or using stationary distributions.
- Combine games $A$ and $B$ into a new game $C$, where you alternate between games $A$ and $B$ with a provided probability. Game $C$ ends up being a winning game.