First Order Theories - Basic Concepts

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22 February 2013
1 Motivation
Outline

1. Motivation

2. Main concepts
Motivation

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(ii) First-order theories formalize the above structures to enable reasoning.
(iii) Fragments of theories may be efficiently decidable.
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Two formulae $F_1$ and $F_2$ are equivalent in theory $T$, or $T$-equivalent, if $T \models F_1 \iff F_2$. In other words, for every $T$-interpretation $I$, we must have, $I \models F_1$ if and only if $I \models F_2$. 
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A theory $T$ is decidable if $T \models F$ is decidable for every $\Sigma$-formula $F$. 
Combination of theories

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**Observation**

FOL is the empty theory, i.e., the theory with no axioms.