First Order Theories - Recursive Data Structures

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Introduction

Theory of Lists
General Theory of RDS
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Introduction

Recursive data structures
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The theory of recursive data structures ($T_{RDS}$) describes a set of data structures such as linked lists, stacks and binary trees that are ubiquitous in programming.
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The theory of recursive data structures ($T_{RDS}$) describes a set of data structures such as linked lists, stacks and binary trees that are ubiquitous in programming. $T_{RDS}$ formalizes reasoning over such structures.
Introduction
Theory of Lists
General Theory of RDS
Theory of Acyclic Lists
Theory of Lists with Specified Atoms
Theory of Lists with Equality

Theory of Lists
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Main points

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where,

(i) $\text{cons}$ is a binary function called the constructor: $\text{cons}(a, b)$ is the list obtained by concatenating $a$ to $b$.

(ii) $\text{car}$ is a unary function, called the left projector: $\text{car}(\text{cons}(a, b)) = a$. 
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(iv) $\text{atom}$ is a unary predicate: $\text{atom}(x)$ is true if and only if $x$ is a single element list.
Theory of Lists

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The theory of lists, $T_{\text{cons}}$, has signature:

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(iv) $\text{atom}$ is a unary predicate: $\text{atom}(x)$ is true if and only if $x$ is a single element list.

(v) $=$ is a binary predicate.
Axiom set of the Theory of Lists
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Subramani First Order Theories
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(A7.) $\forall x \forall y \, \neg \text{atom}(\text{cons}(x, y))$. 
General Theory of RDS
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Main points
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(A4.) $(\forall x_1)(\forall x_2)\ldots(\forall x_n) \pi_i^C(C(x_1, x_2, \ldots, x_n)) = x_i$ for each $i \in \{1, 2, \ldots, n\}$. 
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(A5.) $(\forall x) (\neg \text{atom}_C(x) \rightarrow C(\pi_1^C(x), \pi_2^C(x), \ldots, \pi_n^C(x)) = x)$. 
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Acylic lists
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(i) The theory of acyclic lists, $T_{\text{cons}}^+$, is used to reason about structures such as stacks, which are naturally acyclic.
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(ii) In addition to the axioms of $T_{\text{cons}}$, it has the following axiom schema:
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(ii) In addition to the axioms of $T_{\text{cons}}$, it has the following axiom schema:

$$(\forall x) \: \text{car}(x) \neq x.$$
Main points

(i) The theory of acyclic lists, $T^+_\text{cons}$, is used to reason about structures such as stacks, which are naturally acyclic.

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(A3.) $(\forall x) \text{car}(\text{car}(x)) \neq x.$
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\[(A1.) \ (\forall x) \ car(x) \neq x.\]
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Specifying atomic behavior
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The axioms of $T^+_{\text{cons}}$ do not specify the behavior of cons and cdr on atoms.
Specifying atomic behavior

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The axioms of $T_{\text{cons}}^+$ do not specify the behavior of \texttt{cons} and \texttt{cdr} on atoms. Adding the axiom,

$$(\forall x) \ \text{atom}(x) \rightarrow [\text{atom(car}(x)) \land \text{atom(cdr}(x))]$$

gives a new theory, viz., $T_{\text{cons}}^{\text{atom}}$. 
Theory of Lists with Equality
Theory of Lists with Equality

Main points
Main points

(i) The theory $T_{\text{cons}}^=$, is the theory of lists with equality.
Introduction

Theory of Lists

General Theory of RDS

Theory of Acyclic Lists

Theory of Lists with Specified Atoms

Theory of Lists with Equality

Theory of Lists with Equality

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(i) The theory $T_{\text{cons}} = \text{equal}$, is the theory of lists with equality.

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Example
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Theory of Lists with Equality

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Example
Consider the following formula:
Theory of Lists with Equality

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Example

Consider the following formula:

$$F : [(\text{car}(a) = \text{car}(b)) \land (\text{cdr}(a) = \text{cdr}(b)) \land \neg \text{atom}(a) \land \neg \text{atom}(b)] \rightarrow [f(a) = f(b)]$$
Main points

(i) The theory $T_{\text{cons}}=\text{cons}$, is the theory of lists with equality.

(ii) It is a combination of two theories, viz., the theory of equality and the theory of lists.

(iii) Its signature is $\Sigma_E \cup \Sigma_{\text{cons}}$.

(iv) Its set of axioms is the union of the axiom set of $T_E$ and $T_{\text{cons}}$.

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Is $F$ $T_{\text{cons}}=\text{cons}$-valid?
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