Induction - Complete Induction

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1 Complete Induction
Complete Induction or Strong Induction

Axiom Schema

\[ (\forall n) \left( (\forall n') (n' < n \rightarrow P(n')) \rightarrow P(n) \right) \rightarrow (\forall x) P(x). \]

Note: Do we need a base case? Has it been addressed?
Complete Induction or Strong Induction

Axiom Schema

\[ \left( \forall n \right) \left( \forall n' \right) \left( \left( n' < n \right) \rightarrow P(n') \right) \rightarrow P(n) \rightarrow \left( \forall x \right) P(x). \]
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\[ ((\forall n) (\forall n') ((n' < n) \rightarrow P(n')) \rightarrow P(n)] \rightarrow (\forall x) P(x). \]

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\[\left[\left(\forall n \left( \forall n' \left( (n' < n) \rightarrow P(n') \right) \right) \rightarrow P(n)\right] \rightarrow \left(\forall x\right) P(x).\]

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Axiom Schema

\[ ((\forall n) (\forall n') ((n' < n) \rightarrow P(n')) \rightarrow P(n)) \rightarrow (\forall x) P(x). \]

Note

Do we need a base case? Has it been addressed?
Example

Show that every number greater than or equal to 8 can be expressed in the form $5 \cdot a + 3 \cdot b$, for suitably chosen $a$ and $b$. 

Proof.
The conjecture is clearly true for 8, 9 and 10.
Assume that the conjecture holds for all $r$, $8 \leq r \leq k$.
Consider the integer $k+1$.
Without loss of generality, we assume that $(k+1) \geq 11$.
Observe that $(k+1) - 3 = k - 2$ is at least 8 and less than $k$.
As per the inductive hypothesis, $k-2$ can be expressed in the form $3a+5b$, for suitably chosen $a$ and $b$.
It follows that $(k+1) = 3(a+1) + 5b$, can also be so expressed.
Applying the second principle of mathematical induction, we conclude that the conjecture is true.
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The theory $T^*_{PA}$

Consider the theory of $T^*_{PA}$, which is the theory of Peano arithmetic $T_{PA}$, augmented by the following axioms:

- $A_1$: $(\forall x)(\forall y) (x < y) \rightarrow [\text{quot}(x, y) = 0].$

- $A_2$: $(\forall x)(\forall y) (y > 0) \rightarrow [\text{quot}(x + y, y) = \text{quot}(x, y) + 1].$

- $A_3$: $(\forall x)(\forall y) (x < y) \rightarrow [\text{rem}(x, y) = x].$

- $A_4$: $(\forall x)(\forall y) (y > 0) \rightarrow [\text{rem}(x + y, y) = \text{rem}(x, y)].$

Example: Argue that $(\forall x)(\forall y) (y > 0) \rightarrow [\text{rem}(x, y) < y].$
The theory $T^*_PA$ is the theory of Peano arithmetic $T_PA$, augmented by the following axioms:

A1. $\forall x \forall y (x < y) \rightarrow [\text{quot}(x, y) = 0]$.

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The theory $T^*_{PA}$

Peano Arithmetic with division

Consider the theory of $T^*_{PA}$, which is the theory of Peano arithmetic $T_{PA}$, augmented by the following axioms:

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3. $(\forall x)(\forall y) (x < y \rightarrow [\text{rem}(x, y) = x])$
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$A_1$. $(\forall x)(\forall y) (x < y) \rightarrow [\text{quot}(x, y) = \text{rem}(x, y)]$
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Complete Induction

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Example

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$(\forall x)(\forall y) (y > 0) \rightarrow [\text{rem}(x, y) < y]$. 