Feedback Vertex Set Problem: Part 1

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February 18, 2014
The Formulation of the Problem

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**Definition**

A subset of $V$ whose removal from $G$ leaves an acyclic graph, is called a feedback set.
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Definition

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Remark

Observe that any graph possesses a feedback set.
Recalling Some Topics

We need to recall the notion of a field, linear space over a field, and the directed sum of linear spaces.
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You may have a look at any book on General and Linear Algebra.
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You may have a look at any book on General and Linear Algebra. Also wiki provides good source of information where one can recall basic concepts and facts.
Definition

Let $GF(2)$ denote the set $\{0, 1\}$ with the operations $+$ and $\cdot$ defined on its elements by the following rules: $0 + 0 = 0$, $0 + 1 = 1$, $1 + 0 = 1$, $1 + 1 = 0$. 

Definition

For a positive integer $m \geq 1$, let $GF(2)^m$ denote the set of all vectors of length $m$, whose elements are from $GF(2)$. If $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ are two vectors from $GF(2)^m$, then the vector $x + y$ is defined as follows:

$x + y = (x_1 + y_1, \ldots, x_m + y_m)$.

If $\lambda \in GF(2)$ and $x = (x_1, \ldots, x_m) \in GF(2)^m$, then the vector $\lambda \cdot x$ is defined as $\lambda \cdot x = (\lambda \cdot x_1, \ldots, \lambda \cdot x_m)$.

Definition

A non-empty set $L \subseteq GF(2)^m$ is defined to be a linear space over $GF(2)$, if for any $x, y \in L$ one has $x + y \in L$.

Remark

Observe that both of $L = \{0 = (0, \ldots, 0)\}$ and $L = GF(2)^m$ form a linear space over $GF(2)$. 

Vahan Mkrtchyan  
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The Field $GF(2)$ and the vector space $GF(2)^m$

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Vahan Mkrtchyan  
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Observe that both of $L = \{0 = (0, ..., 0)\}$ and $L = GF(2)^m$ form a linear space over $GF(2)$.
Definition

If \( L, L' \subseteq GF(2)^m \) are two linear spaces over \( GF(2) \), and \( L \subseteq L' \), then \( L \) is said to be a subspace of \( L' \).
Linear Subspace, Linear independence and the dimension of a subspace

**Definition**

If $L, L' \subseteq GF(2)^m$ are two linear spaces over $GF(2)$, and $L \subseteq L'$, then $L$ is said to be a subspace of $L'$.

**Definition**

If $x_1, \ldots, x_k$ are $k$ vectors from $GF(2)^m$, then these vectors are said to be linearly independent over $GF(2)$, if for any $\lambda_1, \ldots, \lambda_k \in GF(2)$ the equality

\[ \lambda_1 \cdot x_1 + \ldots + \lambda_k \cdot x_k = 0 = (0, \ldots, 0) \]

implies that $\lambda_1 = \ldots = \lambda_k = 0$. 

Remark

Observe that the dimension of $L = \{0 = (0, \ldots, 0)\}$ is zero, while it can be shown that the dimension of $L = GF(2)^m$ is $m$. 

Vahan Mkrtchyan
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**Definition**

If $L \subseteq GF(2)^m$ is a linear space, then its dimension is defined to be the maximum number of linearly independent vectors from $L$. 

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Vahan Mkrtchyan  
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Observe that the dimension of $L = \{0 = (0, ..., 0)\}$ is zero, while it can be shown that the dimension of $L = GF(2)^m$ is $m$. 
Definition

If $x_1, \ldots, x_k$ are $k$ vectors from $GF(2)^m$, then an expression of the form

$$\lambda_1 \cdot x_1 + \ldots + \lambda_k \cdot x_k$$

is called a linear combination of $x_1, \ldots, x_k$. 

This is in fact the smallest subspace of $GF(2)^m$ that contains the vectors $x_1, \ldots, x_k$. 

Vahan Mkrtchyan
The span of the vectors $x_1, \ldots, x_k$

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**Definition**

The set of all ($= 2^k$) linear combinations of $x_1, \ldots, x_k$ forms a linear space, which is called the span of the vectors $x_1, \ldots, x_k$.
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**Definition**

The set of all ($= 2^k$) linear combinations of $x_1, \ldots, x_k$ forms a linear space, which is called the span of the vectors $x_1, \ldots, x_k$. This is in fact the smallest subspace of $GF(2)^m$ that contains the vectors $x_1, \ldots, x_k$. 
Definition

Let $L, L', L''$ be linear subspaces of $GF(2)^m$. Then $L$ is said to be directed sum of subspaces $L'$ and $L''$, if for each $x \in L$ there exist exactly one $x' \in L'$ and exactly one $x'' \in L''$, such that $x = x' + x''$. This fact will be denoted by $L = L' \oplus L''$.

Theorem

If $L = L' \oplus L''$, then the dimension of $L$ is equal to the sum of dimensions of $L'$ and $L''$. 

Vahan Mkrtchyan

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Vahan Mkrtchyan Feedback Vertex Set Problem: Part 1
The directed some of linear subspaces and their dimension

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Vahan Mkrtchyan

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Let $G$ be a graph, which contains $m$ edges.
The Cycle Space of a Graph

The Characteristic vector of a cycle

Let $G$ be a graph, which contains $m$ edges. Assume that its edges are ordered as follows: $e_1, \ldots, e_m$. 

The Cycle Space

The span of characteristic vectors corresponding to all simple cycles of $G$, is called the cycle space of the graph $G$. 

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Vahan Mkrtchyan

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Let $G$ be a graph, which contains $m$ edges. Assume that its edges are ordered as follows: $e_1, \ldots, e_m$. Let $C$ be a simple cycle of $G$. 

The Characteristic vector of a cycle

Consider the characteristic vector of $C$ defined as follows:

$$\chi_C = (l_1(C), \ldots, l_m(C)),$$

where $l_i(C) = 1$ if $e_i \in C$, and $l_i(C) = 0$ if $e_i \notin C$.

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Vahan Mkrtchyan

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The Dimension of the Cycle Space

**Theorem**

The dimension of the cycle space of a graph \( G = (V, E) \), denoted by \( \text{cyc}(G) \), is given by the formula: \( \text{cyc}(G) = |E| - |V| + \text{comps}(G) \), where \( \text{comps}(G) \) is the number of connected components of \( G \).

**Proof.**

The proof can be found in chapter 6 of Vazirani’s book. It uses the notion of an orthogonal subspace in Euclidean spaces.
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