Discrete Probability - Rudiments, Expectation and Variance

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Outline

1 Preliminaries
   - Sample Space and Events
   - Defining Probabilities on Events
   - Conditional Probability
   - Independent Events
   - Bayes’ Formula
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2 Random Variables
   - Expectation
   - Expectation of a function of a random variable
   - Linearity of Expectation
   - Variance
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4 Variance of some common random variables
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Sample Space and Events

**Definition**

A random experiment is an experiment whose outcome is not known in advance, but belongs to a non-empty, non-singleton set called the sample space (usually denoted by $S$).
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**Example**

(i) Suppose that the experiment consists of tossing a coin. Then, $S = \{H, T\}$.

(ii) Suppose that the experiment consists of tossing a die. Then, $S = \{1, 2, 3, 4, 5, 6\}$.

(iii) Suppose that the experiment consists of tossing two coins. Then, $S = \{HH, HT, TH, TT\}$.

(iv) Suppose that the experiment consists of measuring the life of a battery. Then, $S = (0, \infty)$.
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Definition

Any subset of the sample space $S$ is called an event.
Example events

(i) In the single coin tossing experiment, \{H\} is an event.

(ii) In the die tossing experiment, \{1, 3, 5\} is an event.

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Combining Events

Definition
Given two events $E$ and $F$, the event $E \cup F$ (union) is defined as the event whose outcomes are in $E$ or $F$; e.g., in the die tossing experiment, the union of the events $E = \{2, 4\}$ and $F = \{1\}$ is $\{1, 2, 4\}$.

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Given two events $E$ and $F$, the event $E \cap F$ (intersection) is defined as the event whose outcomes are in $E$ and $F$; e.g., in the die tossing experiment, the intersection of the events $E = \{1, 2, 3\}$ and $F = \{1\}$ is $\{1\}$.

Definition
Given an event $E$, the event $E^c$ (complement) denotes the event whose outcomes are in $S$, but not in $E$; e.g., in the die tossing experiment, the complement of the event $E = \{1, 2, 3\}$ is $\{4, 5, 6\}$.
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Defining Probabilities on Events

Assigning probabilities
Let $S$ denote a sample space. We assume that the number $P(E)$ is assigned to each event $E$ in $S$, such that:

(i) $0 \leq P(E) \leq 1$.
(ii) $P(S) = 1$.
(iii) If $E_1, E_2, \ldots, E_n$ are mutually exclusive events, then,
$$P(E_1 \cup E_2 \cup \ldots \cup E_n) = \sum_{i=1}^{n} P(E_i).$$

$P(E)$ is called the probability of event $E$.

The 2-tuple $(S, P)$ is called a probability space.

Example
In the coin tossing experiment, if we assume that the coin is fair, then $P(\{H\}) = P(\{T\}) = \frac{1}{2}$.
If on the other hand, the coin is biased, then we could have, $P(\{H\}) = \frac{1}{4}$ and $P(\{T\}) = \frac{3}{4}$.

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(i) Let $E$ be an arbitrary event on the sample space $S$. Then,
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(ii) Let $E$ and $F$ denote two arbitrary events on the sample space $S$. Then,
$$P(E \cup F) = P(E) + P(F) - P(EF).$$

What is $P(E \cup F)$, when $E$ and $F$ are mutually exclusive?

Let $G$ be another event on $S$. What is $P(E \cup F \cup G)$?

Exercise
Consider the experiment of tossing two coins and assume that all 4 outcomes are equally likely. Let $E$ denote the event that the first coin turns up heads and $F$ denote the event that the second coin turns up heads. What is the probability that either the first or the second coin turns up heads?
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What is $P(E \cup F)$, when $E$ and $F$ are mutually exclusive?
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Consider the experiment of tossing two coins and assume that all 4 outcomes are equally likely. Let $E$ denote the event that the first coin turns up heads and $F$ denote the event that the second coin turns up heads.
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Note

(i) Let E be an arbitrary event on the sample space S. Then, \( P(E) + P(E^c) = 1 \).

(ii) Let E and F denote two arbitrary events on the sample space S. Then,
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P(E \cup F) = P(E) + P(F) - P(\mathit{EF}).
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What is \( P(E \cup F) \), when E and F are mutually exclusive?
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Motivation

Consider the experiment of tossing two fair coins.
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Consider the experiment of tossing two fair coins. What is the probability that both coins turn up heads?
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Consider the experiment of tossing two fair coins. What is the probability that both coins turn up heads? Now, assume that the first coin turns up heads. 

Definition

Let $E$ and $F$ denote two events on a sample space $S$. The conditional probability of $E$, given that the event $F$ has occurred is denoted by $P(E|F)$ and is equal to $\frac{P(EOF)}{P(F)}$, assuming $P(F) > 0$.

Example

In the previously discussed coin tossing example, let $E$ denote the event that both coins turn up heads and $F$ denote the event that the first coin turns up heads. Accordingly, we are interested in $P(E|F)$.

Observe that $P(F) = \frac{1}{2}$ and $P(EOF) = \frac{1}{4}$. Hence, $P(E|F) = \frac{1}{4} \div \frac{1}{2} = \frac{1}{2}$.

Notice that $P(E) = \frac{1}{4} \neq P(E|F)$.
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Consider the experiment of tossing two fair coins. What is the probability that both coins turn up heads? Now, assume that the first coin turns up heads. What is the probability that both coins turn up heads?
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Let $E$ and $F$ denote two events on a sample space $S$. The conditional probability of $E$, given that the event $F$ has occurred is denoted by $P(E \mid F)$ and is equal to $\frac{P(EF)}{P(F)}$, assuming $P(F) > 0$.

Example

In the previously discussed coin tossing example, let $E$ denote the event that both coins turn up heads and $F$ denote the event that the first coin turns up heads. Accordingly, we are interested in $P(E \mid F)$. Observe that $P(F) = \frac{1}{2}$ and $P(EF) = \frac{1}{4}$. 

Therefore, $P(E \mid F) = \frac{P(EF)}{P(F)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$. 

Notice that $P(E) = \frac{1}{4} \neq P(E \mid F) = \frac{1}{2}$. 

Subramani Probability Theory
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Consider the experiment of tossing two fair coins. What is the probability that both coins turn up heads? Now, assume that the first coin turns up heads. What is the probability that both coins turn up heads?

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Some more examples

Example

A family has two children. What is the conditional probability that both are boys given that at least one of them is a boy? Assume that the sample space is $S = \{(b, g), (b, b), (g, b), (g, g)\}$ and that all outcomes are equally likely.
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Assume that an urn contains 7 black balls and 5 white balls.
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Assume that an urn contains 7 black balls and 5 white balls. Two balls are chosen from this urn, one after the other, without replacement and at random. What is the probability that both balls are black?
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Let $E$ denote the event that the first ball is black and $F$ denote the event that the second ball is black.
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Solution

Let $E$ denote the event that the first ball is black and $F$ denote the event that the second ball is black. Clearly, we are interested in $P(EF)$. 
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Solution

Let \( E \) denote the event that the first ball is black and \( F \) denote the event that the second ball is black. Clearly, we are interested in \( P(EF) \). Observe that \( P(E) = \frac{7}{12} \) and \( P(F \mid E) = \)
Some more examples

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Subramani  Probability Theory
Some more examples

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### Some more examples

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Assume that an urn contains 7 black balls and 5 white balls. Two balls are chosen from this urn, one after the other, without replacement and at random. What is the probability that both balls are black?

#### Solution

Let $E$ denote the event that the first ball is black and $F$ denote the event that the second ball is black. Clearly, we are interested in $P(EF)$. Observe that $P(E) = \frac{7}{12}$ and $P(F | E) = \frac{6}{11}$. Now, $P(F | E) = \frac{P(EF)}{P(E)}$, and hence,

$$P(EF) = P(F | E) \cdot P(E) = \frac{6}{11} \cdot \frac{7}{12} = \frac{42}{132}.$$
Outline

1 Preliminaries
   - Sample Space and Events
   - Defining Probabilities on Events
   - Conditional Probability
   - Independent Events
   - Bayes’ Formula

2 Random Variables
   - Expectation
   - Expectation of a function of a random variable
   - Linearity of Expectation
   - Variance

3 Identities

4 Variance of some common random variables
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Two events $E$ and $F$ on a sample space $S$ are said to be independent, if the occurrence of one does not affect the occurrence of the other.
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Alternatively,

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Exercise

Consider the experiment of tossing two fair dice. Let $F$ denote the event that the first die turns up $4$. Let $E_1$ denote the event that the sum of the faces of the two dice is $6$. Let $E_2$ denote the event that the sum of the faces of the two dice is $7$. Are $E_1$ and $F$ independent? How about $E_2$ and $F$?
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Two events $E$ and $F$ on a sample space $S$ are said to be independent, if the occurrence of one does not affect the occurrence of the other. Mathematically,

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Bayes’ Formula

Derivation

Let $E$ and $F$ denote two arbitrary events on a sample space $S$. 

Clearly, $E = EF \cup EF^c$, where the events $EF$ and $EF^c$ are mutually exclusive.

Now, observe that,

$$P(E) = P(EF) + P(EF^c) = P(E|F)P(F) + P(E|F^c)(1 - P(F))$$

Thus, the probability of an event $E$ is the weighted average of the conditional probability of $E$, given that event $F$ has occurred and the conditional probability of $E$, given that event $F$ has not occurred, each conditional probability being given as much weight as the probability of the event that it is conditioned on, has of occurring.
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$$= P(E \mid F)P(F) + P(E \mid F^c)P(F^c)$$
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Example

Consider two urns.

Urn 1 has 2 white balls and 7 black balls.
Urn 2 has 5 white balls and 6 black balls.
A fair coin is tossed.
If the coin turns up heads, a ball is drawn from Urn 1, otherwise, a ball is drawn from Urn 2.
Given that the ball drawn was white, what is the conditional probability that it was drawn from Urn 1?

Solution
Let \( W \) denote the event that a white ball was drawn and let \( H \) denote the event that the coin turned up heads.
(Note that \( H \) is precisely the event that the ball was drawn from Urn 1.)
We are therefore interested in the quantity \( P(H|W) \).
From conditional probability, we know that, 
\[
P(H|W) = \frac{P(HW)}{P(W)}.
\]
\( P(HW) = P(W|H) \cdot P(H) \)
\[
= \frac{2}{9} \cdot \frac{1}{2} = \frac{1}{9}.
\]
As per Bayes' formula, 
\[
P(W) = P(W|H) \cdot P(H) + P(W|H^c)(1 - P(H))
\]
\[
= \frac{2}{9} \cdot \frac{1}{2} + \frac{5}{11} \cdot \frac{1}{2} = \frac{67}{198}.
\]
Therefore, 
\[
P(H|W) = \frac{1/9}{67/198} = \frac{22}{67},
\]
i.e., the conditional probability that the ball was drawn from Urn 1, given that it is white, is \( \frac{22}{67} \).
One Final Example

Example

Consider two urns. Urn 1 has 2 white balls and 7 black balls.
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Example

Consider two urns. Urn 1 has 2 white balls and 7 black balls. Urn 2 has 5 white balls and 6 black balls.

Let $W$ denote the event that a white ball was drawn and let $H$ denote the event that the coin turned up heads. (Note that $H$ is precisely the event that the ball was drawn from Urn 1.) We are therefore interested in the quantity $P(H|W)$.

From conditional probability, we know that,

$$P(H|W) = \frac{P(HW)}{P(W)}.$$

$$P(HW) = P(W|H) \cdot P(H) = \frac{2}{9} \cdot \frac{1}{2} = \frac{1}{9}.$$

As per Bayes' formula,

$$P(W) = P(W|H) \cdot P(H) + P(W|H^c) \cdot (1 - P(H)) = \frac{2}{9} \cdot \frac{1}{2} + \frac{5}{11} \cdot \frac{1}{2} = \frac{67}{198}.$$

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Consider two urns. Urn 1 has 2 white balls and 7 black balls. Urn 2 has 5 white balls and 6 black balls. A fair coin is tossed. If the coin turns up heads, a ball is drawn from Urn 1, otherwise, a ball is drawn from Urn 2. Given that the ball drawn was white, what is the conditional probability that it was drawn from Urn 1?

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From conditional probability, we know that,

\[
P(H|W) = \frac{P(H \cap W)}{P(W)}.
\]

\( P(H \cap W) = P(W|H) \cdot P(H) \)

\( = \frac{2}{9} \cdot \frac{1}{2} = \frac{1}{9} \).

As per Bayes' formula,

\[
P(W) = P(W|H) \cdot P(H) + P(W|H^c) \cdot (1 - P(H)) = \frac{2}{9} \cdot \frac{1}{2} + \frac{5}{11} \cdot \frac{1}{2} = \frac{67}{198}.
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Therefore,

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Solution

Let $W$ denote the event that a white ball was drawn and let $H$ denote the event that the coin turned up heads. (Note that $H$ is precisely the event that the ball was drawn from Urn 1.)

We are therefore interested in the quantity $P(H \mid W)$. 

Let $P(W)$ denote the probability of drawing a white ball. Given that the coin turned up heads, the probability of drawing a white ball is $P(W \mid H)$. If the coin turns up heads, the probability of drawing a white ball is higher from Urn 1, which has a higher proportion of white balls.

Let $P(H)$ denote the probability of the coin turning up heads, which is $0.5$ since the coin is fair.

Let $P(Hc)$ denote the probability of not turning up heads, which is $0.5$.

By Bayes' Theorem,

$$P(H \mid W) = \frac{P(W \mid H) \cdot P(H)}{P(W)}$$

where

- $P(W \mid H) = \frac{2}{9}$ because the probability of drawing a white ball from Urn 1 is $\frac{2}{9}$.
- $P(H) = 0.5$.

First, we calculate $P(W)$:

$$P(W) = P(W \mid H) \cdot P(H) + P(W \mid Hc) \cdot P(Hc)$$

For Urn 1, the probability of drawing a white ball is $\frac{2}{9}$, and for Urn 2, it is $\frac{5}{11}$. Therefore,

$$P(W) = \frac{2}{9} \cdot 0.5 + \frac{5}{11} \cdot 0.5$$

$$P(W) = \frac{1}{9} + \frac{5}{22} = \frac{22}{198} + \frac{45}{198} = \frac{67}{198}$$

Now, we can calculate $P(H \mid W)$:

$$P(H \mid W) = \frac{\frac{2}{9} \cdot 0.5}{\frac{67}{198}} = \frac{\frac{1}{9}}{\frac{67}{198}} = \frac{1}{9} \cdot \frac{198}{67} = \frac{22}{67}$$

Therefore, the conditional probability that the ball was drawn from Urn 1, given that it is white, is $\frac{22}{67}$. 

This example illustrates the application of Bayes' Theorem in a practical scenario, highlighting how prior probabilities and conditional probabilities interact to provide updated probabilities based on new information.
Consider two urns. Urn 1 has 2 white balls and 7 black balls. Urn 2 has 5 white balls and 6 black balls. A fair coin is tossed. If the coin turns up heads, a ball is drawn from Urn 1, otherwise, a ball is drawn from Urn 2. Given that the ball drawn was white, what is the conditional probability that it was drawn from Urn 1?

Let $W$ denote the event that a white ball was drawn and let $H$ denote the event that the coin turned up heads. (Note that $H$ is precisely the event that the ball was drawn from Urn 1.)

We are therefore interested in the quantity $P(H \mid W)$. From conditional probability, we know that, $P(H \mid W) = \frac{P(HW)}{P(W)}$. 

Therefore, $P(H \mid W) = \frac{1}{9}$, i.e., the conditional probability that the ball was drawn from Urn 1, given that it is white, is $\frac{22}{67}$. 


One Final Example

Example

Consider two urns. Urn 1 has 2 white balls and 7 black balls. Urn 2 has 5 white balls and 6 black balls. A fair coin is tossed. If the coin turns up heads, a ball is drawn from Urn 1, otherwise, a ball is drawn from Urn 2. Given that the ball drawn was white, what is the conditional probability that it was drawn from Urn 1?

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Let $W$ denote the event that a white ball was drawn and let $H$ denote the event that the coin turned up heads. (Note that $H$ is precisely the event that the ball was drawn from Urn 1.) We are therefore interested in the quantity $P(H \mid W)$. From conditional probability, we know that, $P(H \mid W) = \frac{P(HW)}{P(W)}$.

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Example

Consider two urns. Urn 1 has 2 white balls and 7 black balls. Urn 2 has 5 white balls and 6 black balls. A fair coin is tossed. If the coin turns up heads, a ball is drawn from Urn 1, otherwise, a ball is drawn from Urn 2. Given that the ball drawn was white, what is the conditional probability that it was drawn from Urn 1?

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$P(HW) = P(W \mid H) \cdot P(H) = \frac{2}{9} \cdot \frac{1}{2} = \frac{1}{9}$. 
Example

Consider two urns. Urn 1 has 2 white balls and 7 black balls. Urn 2 has 5 white balls and 6 black balls. A fair coin is tossed. If the coin turns up heads, a ball is drawn from Urn 1, otherwise, a ball is drawn from Urn 2. Given that the ball drawn was white, what is the conditional probability that it was drawn from Urn 1?

Solution

Let $W$ denote the event that a white ball was drawn and let $H$ denote the event that the coin turned up heads. (Note that $H$ is precisely the event that the ball was drawn from Urn 1.)

We are therefore interested in the quantity $P(H \mid W)$. From conditional probability, we know that, $P(H \mid W) = \frac{P(HW)}{P(W)}$.

$P(HW) = P(W \mid H) \cdot P(H) = \frac{2}{9} \cdot \frac{1}{2} = \frac{1}{9}$. As per Bayes' formula,

$$P(W) = \ldots$$
Example

Consider two urns. Urn 1 has 2 white balls and 7 black balls. Urn 2 has 5 white balls and 6 black balls. A fair coin is tossed. If the coin turns up heads, a ball is drawn from Urn 1, otherwise, a ball is drawn from Urn 2. Given that the ball drawn was white, what is the conditional probability that it was drawn from Urn 1?

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$P(HW) = P(W \mid H) \cdot P(H) = \frac{2}{9} \cdot \frac{1}{2} = \frac{1}{9}$. As per Bayes' formula,

$$P(W) = P(W \mid H) \cdot P(H)$$
One Final Example

Example

Consider two urns. Urn 1 has 2 white balls and 7 black balls. Urn 2 has 5 white balls and 6 black balls. A fair coin is tossed. If the coin turns up heads, a ball is drawn from Urn 1, otherwise, a ball is drawn from Urn 2. Given that the ball drawn was white, what is the conditional probability that it was drawn from Urn 1?

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$P(HW) = P(W \mid H) \cdot P(H) = \frac{2}{9} \cdot \frac{1}{2} = \frac{1}{9}$. As per Bayes’ formula,

$$P(W) = P(W \mid H) \cdot P(H) + P(W \mid H^C)(1 - P(H))$$
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Consider two urns. Urn 1 has 2 white balls and 7 black balls. Urn 2 has 5 white balls and 6 black balls. A fair coin is tossed. If the coin turns up heads, a ball is drawn from Urn 1, otherwise, a ball is drawn from Urn 2. Given that the ball drawn was white, what is the conditional probability that it was drawn from Urn 1?

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Let \( W \) denote the event that a white ball was drawn and let \( H \) denote the event that the coin turned up heads. (Note that \( H \) is precisely the event that the ball was drawn from Urn 1.)
We are therefore interested in the quantity \( P(H \mid W) \). From conditional probability, we know that, \( P(H \mid W) = \frac{P(HW)}{P(W)} \).
\[
P(HW) = P(W \mid H) \cdot P(H) = \frac{2}{9} \cdot \frac{1}{2} = \frac{1}{9}.
\]
As per Bayes’ formula,
\[
P(W) = P(W \mid H) \cdot P(H) + P(W \mid H^C)(1 - P(H))
= \frac{2}{9} \cdot \frac{1}{2} + \frac{5}{11} \cdot \frac{1}{2}
\]
\[
= \frac{2}{9} \cdot \frac{1}{2} + \frac{5}{11} \cdot \frac{1}{2}
\]
Therefore, \( P(H \mid W) = \frac{1}{9} \cdot \frac{2}{198} = \frac{2}{67} \), i.e., the conditional probability that the ball was drawn from Urn 1, given that it is white, is \( \frac{2}{67} \).
**Example**

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P(W) = P(W \mid H) \cdot P(H) + P(W \mid H^c)(1 - P(H))
\]

\[
= \frac{2}{9} \cdot \frac{1}{2} + \frac{5}{11} \cdot \frac{1}{2}
\]

\[
= \frac{67}{198}
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$$P(W) = P(W \mid H) \cdot P(H) + P(W \mid H^c)(1 - P(H)) = \frac{2}{9} \cdot \frac{1}{2} + \frac{5}{11} \cdot \frac{1}{2} = \frac{67}{198}$$

Therefore, $P(H \mid W) = \frac{\frac{1}{9}}{\frac{67}{198}} = \frac{22}{67}$.
Example

Consider two urns. Urn 1 has 2 white balls and 7 black balls. Urn 2 has 5 white balls and 6 black balls. A fair coin is tossed. If the coin turns up heads, a ball is drawn from Urn 1, otherwise, a ball is drawn from Urn 2. Given that the ball drawn was white, what is the conditional probability that it was drawn from Urn 1?

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$$P(W) = P(W \mid H) \cdot P(H) + P(W \mid H^C)(1 - P(H))$$

$$= \frac{2}{9} \cdot \frac{1}{2} + \frac{5}{11} \cdot \frac{1}{2}$$

$$= \frac{67}{198}$$

Therefore, $P(H \mid W) = \frac{1}{\frac{67}{198}} = \frac{22}{67}$. 
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Therefore, $P(H \mid W) = \frac{\frac{1}{9}}{\frac{67}{198}} = \frac{22}{67}$, i.e., the conditional probability that the ball was drawn from Urn 1, given that it is white, is $\frac{22}{67}$.
Motivation

In case of certain random experiments, we are not so much interested in the actual outcome, but in some function of the outcome, e.g., in the experiment of tossing two dice, we could be interested in knowing whether or not the sum of the upturned faces is 7. We may not care whether the actual outcome is \((1, 6)\), \((6, 1)\), or . . . .

Example: Let \(X\) denote the random variable that is defined as the sum of two fair dice. What are the values that \(X\) can take?

\[
P\{X = 1\} = 0
\]

\[
P\{X = 2\} = \frac{1}{36}
\]

... 

\[
P\{X = 12\} = \frac{1}{36}
\]
Motivation

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Let $X$ denote the random variable that is defined as the sum of two fair dice.
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Let $X$ denote the random variable that is defined as the sum of two fair dice. What are the values that $X$ can take?
Random Variables

Motivation

In case of certain random experiments, we are not so much interested in the actual outcome, but in some function of the outcome, e.g., in the experiment of tossing two dice, we could be interested in knowing whether or not the sum of the upturned faces is 7. We may not care whether the actual outcome is (1, 6), (6, 1), or . . . .

Example

Let $X$ denote the random variable that is defined as the sum of two fair dice. What are the values that $X$ can take?

$$P\{X = 1\} =$$
Motivation

In case of certain random experiments, we are not so much interested in the actual outcome, but in some function of the outcome, e.g., in the experiment of tossing two dice, we could be interested in knowing whether or not the sum of the upturned faces is 7. We may not care whether the actual outcome is (1, 6), (6, 1), or ....

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P\{X = 2\} = \frac{1}{36} \\
\vdots
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Random Variables

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In case of certain random experiments, we are not so much interested in the actual outcome, but in some function of the outcome, e.g., in the experiment of tossing two dice, we could be interested in knowing whether or not the sum of the upturned faces is 7. We may not care whether the actual outcome is (1, 6), (6, 1), or ....

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P\{X = 2\} = \frac{1}{36}
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\[
\vdots
\]
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P\{X = 12\} = \frac{1}{36}
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Example

Consider the experiment of tossing two fair coins; let $Y$ denote the random variable that counts the number of heads.

$P\{Y = 0\} = \frac{1}{4}$

$P\{Y = 1\} = \frac{1}{2}$

$P\{Y = 2\} = \frac{1}{4}$
Example

Consider the experiment of tossing two fair coins; let \( Y \) denote the random variable that counts the number of heads. What values can \( Y \) take?
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$$P\{Y = 0\} = \frac{1}{4}$$

$$P\{Y = 1\} = \frac{1}{2}$$

$$P\{Y = 2\} = \quad \text{(to be determined)}$$
Example

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\[
\begin{align*}
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\end{align*}
\]

**Definition**

A random variable that can take on only a countable number of possible values is said to be *discrete*. 

---

**Example**

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- $P\{Y = 0\} = \frac{1}{4}$
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- $P\{Y = 2\} = \frac{1}{4}$

Definition

A random variable that can take on only a countable number of possible values is said to be discrete. For a discrete random variable $X$, the probability mass function (pmf) $p(a)$ is defined as:

$$p(a) = P\{X = a\}.$$
The Bernoulli Random Variable

Main idea

Consider an experiment which has exactly two outcomes;
The Bernoulli Random Variable

Main idea

Consider an experiment which has exactly two outcomes; one is labeled a “success” and the other a “failure”.

\[ p(1) = P\{X = 1\} = p \]
\[ p(0) = P\{X = 0\} = 1 - p \]

where \( 0 \leq p \leq 1 \) is the probability that the experiment results in a success.
The Bernoulli Random Variable

Main idea

Consider an experiment which has exactly two outcomes; one is labeled a “success” and the other a “failure”. If we let the random variable $X$ assume the value 1, if the experiment was a success and 0, if the experiment was a failure, then $X$ is said to be a Bernoulli random variable.
The Bernoulli Random Variable

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Consider an experiment which has exactly two outcomes; one is labeled a “success” and the other a “failure”. If we let the random variable \( X \) assume the value 1, if the experiment was a success and 0, if the experiment was a failure, then \( X \) is said to be a Bernoulli random variable. The probability mass function of \( X \) is given by:

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p(1) = P\{X = 1\} = p
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The Bernoulli Random Variable

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p(1) &= P\{X = 1\} = p \\
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\end{align*}
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where \( 0 \leq p \leq 1 \) is the probability that the experiment results in a success.
The Binomial Random Variable

Motivation
Consider an experiment which consists of $n$ independent Bernoulli trials, with the probability of success in each trial being $p$. If $X$ is the random variable that counts the number of successes in the $n$ trials, then $X$ is said to be a Binomial Random Variable.
The Binomial Random Variable

Motivation

Consider an experiment which consists of \( n \) independent Bernoulli trials, with the probability of success in each trial being \( p \). If \( X \) is the random variable that counts the number of successes in the \( n \) trials, then \( X \) is said to be a Binomial Random Variable. The probability mass function of \( X \) is given by:

\[
p(i) = \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i}, \quad i = 0, 1, 2, \ldots, n
\]
The Binomial Random Variable

Motivation

Consider an experiment which consists of $n$ independent Bernoulli trials, with the probability of success in each trial being $p$. If $X$ is the random variable that counts the number of successes in the $n$ trials, then $X$ is said to be a Binomial Random Variable. The probability mass function of $X$ is given by:

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\[
p(i) = P\{X = i\} = C(n, i) \cdot p^i \cdot (1 - p)^{n-i}, \quad i = 0, 1, 2, \ldots n
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The Binomial Random Variable

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Consider an experiment which consists of \( n \) independent Bernoulli trials, with the probability of success in each trial being \( p \). If \( X \) is the random variable that counts the number of successes in the \( n \) trials, then \( X \) is said to be a Binomial Random Variable. The probability mass function of \( X \) is given by:

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\]

Example
Consider the experiment of tossing four fair coins.
The Binomial Random Variable

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Consider an experiment which consists of $n$ independent Bernoulli trials, with the probability of success in each trial being $p$. If $X$ is the random variable that counts the number of successes in the $n$ trials, then $X$ is said to be a Binomial Random Variable. The probability mass function of $X$ is given by:

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Example
Consider the experiment of tossing four fair coins. What is the probability that you will get two heads and two tails?
The Binomial Random Variable

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Consider an experiment which consists of $n$ independent Bernoulli trials, with the probability of success in each trial being $p$. If $X$ is the random variable that counts the number of successes in the $n$ trials, then $X$ is said to be a Binomial Random Variable. The probability mass function of $X$ is given by:

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Example

Consider the experiment of tossing four fair coins. What is the probability that you will get two heads and two tails?
Example (contd.)

Solution

Let the event of heads turning up denote a “success.”
Example (contd.)

Solution

Let the event of heads turning up denote a “success.” Accordingly, we are interested in the probability of getting exactly two successes in four Bernoulli trials.
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Let the event of heads turning up denote a “success.” Accordingly, we are interested in the probability of getting exactly two successes in four Bernoulli trials. As discussed above,

\[ p(2) = \]
Solution

Let the event of heads turning up denote a “success.” Accordingly, we are interested in the probability of getting exactly two successes in four Bernoulli trials. As discussed above,

\[ p(2) = \binom{4}{2} \cdot \left( \frac{1}{2} \right)^2 \]
Example (contd.)

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Let the event of heads turning up denote a “success.” Accordingly, we are interested in the probability of getting exactly two successes in four Bernoulli trials. As discussed above,

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Example (contd.)

Solution

Let the event of heads turning up denote a “success.” Accordingly, we are interested in the probability of getting exactly two successes in four Bernoulli trials. As discussed above,

\[
p(2) = C(4, 2) \cdot \left(\frac{1}{2}\right)^2 \cdot \left(1 - \frac{1}{2}\right)^2
\]

\[
= \frac{3}{8}
\]
The Geometric Random Variable

Motivation

Suppose that independent Bernoulli trials, each with probability $p$ of success are performed until a success occurs.
Motivation

Suppose that independent Bernoulli trials, each with probability $p$ of success are performed until a success occurs. If $X$ is the random variable that counts the number of trials until the first success, then $X$ is said to be a geometric random variable.
The Geometric Random Variable

Motivation
Suppose that independent Bernoulli trials, each with probability $p$ of success are performed until a success occurs. If $X$ is the random variable that counts the number of trials until the first success, then $X$ is said to be a geometric random variable. The probability mass function of $X$ is given by:

$$p(i) = P\{X = i\} = (1 - p)^{i-1} \cdot p,$$
The Geometric Random Variable

Motivation

Suppose that independent Bernoulli trials, each with probability $p$ of success are performed until a success occurs. If $X$ is the random variable that counts the number of trials until the first success, then $X$ is said to be a geometric random variable. The probability mass function of $X$ is given by:

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Outline

1 Preliminaries
   - Sample Space and Events
   - Defining Probabilities on Events
   - Conditional Probability
   - Independent Events
   - Bayes’ Formula

2 Random Variables
   - Expectation
     - Expectation of a function of a random variable
     - Linearity of Expectation
     - Variance

3 Identities

4 Variance of some common random variables
Definition

Let $X$ denote a discrete random variable with probability mass function $p(x)$. The expected value of $X$, denoted by $E[X]$, is defined by:

$$E[X] = \sum_{x} x \cdot p(x)$$

Note $E[X]$ is the weighted average of the possible values that $X$ can assume, each value being weighted by the probability that $X$ assumes that value.

Example

Let $X$ denote the random variable that records the outcome of tossing a fair die. What is $E[X]$?
**Definition**

Let $X$ denote a discrete random variable with probability mass function $p(x)$. 

Expectation

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Let $X$ denote the random variable that records the outcome of tossing a fair die. What is $E[X]$?
Expectation of a Bernoulli Random Variable

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Let $X$ denote a Bernoulli Random Variable with $p$ denoting the probability of success.
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Let $X$ denote a Bernoulli Random Variable with $p$ denoting the probability of success. What is $E[X]$?

Solution:

$E[X] = 1 \cdot p + 0 \cdot (1 - p) = p$
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Let $X$ denote a Bernoulli Random Variable with $p$ denoting the probability of success. What is $E[X]$?

**Solution:**

$$E[X] = 1 \cdot p + 0 \cdot (1 - p)$$
Example

Let $X$ denote a Bernoulli Random Variable with $p$ denoting the probability of success. What is $E[X]$?

**Solution:**

$$E[X] = 1 \cdot p + 0 \cdot (1 - p) = p$$
Expectation of a Binomial Random Variable

Example

Let $X$ denote a Binomial Random Variable with parameters $n$ and $p$. 

Solution:

$E[X] = \sum_{i=0}^{n} i \cdot p^i (1-p)^{n-i}$, by definition

$= n \sum_{i=1}^{n} i \cdot \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i}$

$= n \sum_{i=1}^{n} \frac{n!}{i!(n-i)!} \cdot p^i \cdot (1-p)^{n-i}$

$= n \cdot p \sum_{i=1}^{n} \frac{(n-1)!}{(i-1)!(n-i)!} \cdot p^{i-1} \cdot (1-p)^{n-i}$

$= n \cdot p \sum_{i=1}^{n} \frac{(n-1)!}{(i-1)!(n-i)!} \cdot p^{i-1} \cdot (1-p)^{n-i} - 1 \cdot p \sum_{i=1}^{n} \frac{(n-1)!}{(i-1)!(n-i)!} \cdot p^{i-1} \cdot (1-p)^{n-i}$

$= n \cdot p \sum_{i=1}^{n} \frac{(n-1)!}{(i-1)!(n-i)!} \cdot p^{i-1} \cdot (1-p)^{n-i} - 1 \cdot p \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} \cdot p^i \cdot (1-p)^{n-i}$

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$= n \cdot p \sum_{i=1}^{n} \frac{(n-1)!}{(i-1)!(n-i)!} \cdot p^{i-1} \cdot (1-p)^{n-i} - 1 \cdot p \cdot n \cdot p$
Example

Let $X$ denote a Binomial Random Variable with parameters $n$ and $p$. What is $E[X]$?
Example

Let $X$ denote a Binomial Random Variable with parameters $n$ and $p$. What is $E[X]$?

**Solution:**

$$E[X] = \sum_{i=0}^{n} i \cdot p(i), \text{ by definition}$$
Example

Let $X$ denote a Binomial Random Variable with parameters $n$ and $p$. What is $E[X]$?

Solution:

$$E[X] = \sum_{i=0}^{n} i \cdot p(i), \text{ by definition}$$

$$= \sum_{i=0}^{n} i \cdot C(n, i) \cdot p^i \cdot (1 - p)^{n-i}$$
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**Solution:**

$$E[X] = \sum_{i=0}^{n} i \cdot p(i), \text{ by definition}$$

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Solution:

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E[X] = \sum_{i=0}^{n} i \cdot p^i, \quad \text{by definition}
\]

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= \sum_{i=0}^{n} i \cdot \binom{n}{i} \cdot p^i \cdot (1 - p)^{n-i}
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Expectation of a Binomial Random Variable

Example

Let $X$ denote a Binomial Random Variable with parameters $n$ and $p$. What is $E[X]$?

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$$= n \cdot p \sum_{i=1}^{n} \frac{(n-1)!}{(i-1)!(n-i)!} \cdot p^{i-1} \cdot (1 - p)^{n-i}$$
Example

Let $X$ denote a Binomial Random Variable with parameters $n$ and $p$. What is $E[X]$?

**Solution:**

\[
E[X] = \sum_{i=0}^{n} i \cdot p(i), \text{ by definition}
\]
\[
= \sum_{i=0}^{n} i \cdot C(n, i) \cdot p^i \cdot (1 - p)^{n-i}
\]
\[
= \sum_{i=0}^{n} i \cdot \frac{n!}{i!(n-i)!} \cdot p^i \cdot (1 - p)^{n-i}
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Example

Substituting $k = i - 1$, we get,

$$E[X] = n \cdot p \sum_{k=0}^{n-1} \frac{(n-1)!}{k! \cdot (n - k - 1)!} \cdot p^k \cdot (1-p)^{n-k-1}$$
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$$E[X] = n \cdot p \sum_{k=0}^{n-1} \frac{(n - 1)!}{k! \cdot (n - k - 1)!} \cdot p^k \cdot (1 - p)^{n-k-1}$$

$$= n \cdot p \sum_{k=0}^{n-1} \frac{(n - 1)!}{k! \cdot ((n - 1) - k)!} \cdot p^k \cdot (1 - p)^{(n-1)-k}$$
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Substituting \( k = i - 1 \), we get,

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E[X] = n \cdot p \sum_{k=0}^{n-1} \frac{(n-1)!}{k! \cdot (n-k-1)!} \cdot p^k \cdot (1-p)^{n-k-1}
\]

\[
= n \cdot p \sum_{k=0}^{n-1} \frac{(n-1)!}{k! \cdot ((n-1)-k)!} \cdot p^k \cdot (1-p)^{(n-1)-k}
\]

\[
= n \cdot p \sum_{k=0}^{n-1} C(n-1, k) \cdot p^k \cdot (1-p)^{(n-1)-k}
\]

Subramani Probability Theory
Expectation of a Binomial Random Variable (contd.)

Example

Substituting $k = i - 1$, we get,

$$E[X] = n \cdot p \sum_{k=0}^{n-1} \frac{(n - 1)!}{k! \cdot (n - 1 - k)!} \cdot p^k \cdot (1 - p)^{n-k-1}$$

$$= n \cdot p \sum_{k=0}^{n-1} \frac{(n - 1)!}{k! \cdot ((n - 1) - k)!} \cdot p^k \cdot (1 - p)^{(n-1)-k}$$

$$= n \cdot p \sum_{k=0}^{n-1} C(n - 1, k) \cdot p^k \cdot (1 - p)^{(n-1)-k}$$

$$= n \cdot p \cdot [p + (1 - p)]^{n-1}, \text{ Binomial theorem}$$
Example

Substituting $k = i - 1$, we get,

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E[X] = \sum_{k=0}^{n-1} \frac{(n-1)!}{k! \cdot (n-k-1)!} \cdot p^k \cdot (1-p)^{n-k-1}
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= \sum_{k=0}^{n-1} C(n-1, k) \cdot p^k \cdot (1-p)^{(n-1)-k}
\]

\[
= n \cdot p \cdot [p + (1-p)]^{n-1}, \text{ Binomial theorem}
\]

\[
= n \cdot p \cdot 1
\]
Preliminaries
Random Variables
Identities
Variance of some common random variables

Expectation
Expectation of a function of a random variable
Linearity of Expectation
Variance

Expectation of a Binomial Random Variable (contd.)

Example

Substituting \( k = i - 1 \), we get,

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E[X] = n \cdot p \sum_{k=0}^{n-1} \frac{(n-1)!}{k! \cdot (n-k-1)!} \cdot p^k \cdot (1-p)^{n-k-1}
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= n \cdot p \sum_{k=0}^{n-1} \frac{(n-1)!}{k! \cdot ((n-1)-k)!} \cdot p^k \cdot (1-p)^{(n-1)-k}
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= n \cdot p \sum_{k=0}^{n-1} C(n-1, k) \cdot p^k \cdot (1-p)^{(n-1)-k}
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\[
= n \cdot p \cdot [p + (1-p)]^{n-1}, \text{ Binomial theorem}
\]

\[
= n \cdot p \cdot 1
\]

\[
= n \cdot p
\]
## Expectation of a Geometric Random Variable

### Example

Let $X$ denote a Geometric Random Variable with parameters $n$ and $p$. 

By definition, the expectation $E[X]$ can be calculated as:

$$E[X] = \sum_{i=1}^{\infty} i \cdot p_i(1-p)^{i-1} \cdot p = \sum_{i=1}^{\infty} i \cdot q^{i-1} \cdot p,$$

where $q = 1-p$. Therefore,

$$E[X] = p \cdot \sum_{i=1}^{\infty} i \cdot q^{i-1} = p \cdot \frac{1}{(1-q)^2}.$$
Example

Let $X$ denote a Geometric Random Variable with parameters $n$ and $p$. What is $E[X]$?

\[
E[X] = \sum_{i=1}^{\infty} i \cdot p \cdot (1-p)^{i-1},
\]

by definition.

\[
= \sum_{i=1}^{\infty} i \cdot q^{i-1} \cdot p,
\]

where $q = 1 - p$.

\[
= p \cdot \sum_{i=1}^{\infty} i \cdot q^{i-1},
\]

\[
\square
\]
### Expectation of a Geometric Random Variable

**Example**

Let $X$ denote a Geometric Random Variable with parameters $n$ and $p$. What is $E[X]$?

**Solution:**

$$E[X] = \sum_{i=1}^{\infty} i \cdot p(i), \text{ by definition}$$
Example

Let $X$ denote a Geometric Random Variable with parameters $n$ and $p$. What is $E[X]$?

Solution:

$$E[X] = \sum_{i=1}^{\infty} i \cdot p(i), \text{ by definition}$$

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Let $X$ denote a Geometric Random Variable with parameters $n$ and $p$. What is $E[X]$?

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E[X] = \sum_{i=1}^{\infty} i \cdot p(i), \text{ by definition}
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Expectation of a Geometric Random Variable

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$$= p \cdot \sum_{i=1}^{\infty} i \cdot q^{i-1}$$
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Let $X$ denote a Geometric Random Variable with parameters $n$ and $p$. What is $E[X]$?

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$$= p \cdot \sum_{i=1}^{\infty} i \cdot q^{i-1}$$

$$= p \cdot \sum_{i=1}^{\infty} \frac{d}{dq} [q^i]$$
Example

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Solution:

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$$= p \cdot \sum_{i=1}^{\infty} \frac{d}{dq} [q^i]$$

□
Example

Solution:

\[ E[X] = p \cdot \frac{d}{dq} \left[ \sum_{i=1}^{\infty} q^i \right] \]
Expectation of a Geometric Random Variable (contd.)

Example

Solution:

\[ E[X] = p \cdot \frac{d}{dq} \left[ \sum_{i=1}^{\infty} q^i \right] = p \cdot \frac{d}{dq} \left[ \frac{q}{1 - q} \right] \]
Expectation of a Geometric Random Variable (contd.)

Example

Solution:

\[ E[X] = p \cdot \frac{d}{dq} \left[ \sum_{i=1}^{\infty} q^i \right] \]
\[ = p \cdot \frac{d}{dq} \left[ \frac{q}{1 - q} \right] \]
\[ = p \cdot \frac{(1 - q)}{dq} [q] - q \cdot \frac{d}{dq} [1 - q] \]
\[ = p \cdot \frac{(1 - q)}{(1 - q)^2} \]
\[ = p \cdot \frac{1}{1 - q} \]
\[ \square \]
Example

Solution:

\[ E[X] = p \cdot \frac{d}{dq} \left[ \sum_{i=1}^{\infty} q^i \right] \]

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\[ = p \cdot \frac{(1 - q) \cdot \frac{d}{dq} [q] - q \cdot \frac{d}{dq} [1 - q]}{(1 - q)^2} \]

\[ = p \cdot \frac{(1 - q) \cdot 1 - q \cdot (-1)}{(1 - q)^2} \]
Expectation of a Geometric Random Variable (contd.)

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Expectation of a Geometric Random Variable (contd.)

Example

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E[X] = p \cdot \frac{d}{dq} \left[ \sum_{i=1}^{\infty} q^i \right]
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Expectation of a Geometric Random Variable (contd.)

Example

Solution:

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E[X] = p \cdot \frac{d}{dq} \left\{ \sum_{i=1}^{\infty} q^i \right\}
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= p \cdot \frac{1}{(1 - q)^2}
\]

\[
= p \cdot \frac{1}{p^2}
\]

\[
= \frac{1}{p}
\]

\[\square\]
Note

Often times, we are interested in a function of the random variable, rather than the random variable itself. For instance, in the coin-tossing experiment, we could be interested in the square of the number of successes. The question of interest then is how to determine the expectation of a function of a random variable, given that we only know the distribution of the random variable.

Example

Let $X$ be a random variable, with the following pmf:

- $p(0) = 0.3$,
- $p(1) = 0.5$,
- $p(2) = 0.2$.

Compute $E[X^2]$. 

Subramani Probability Theory
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Let $X$ be a random variable, with the following pmf:

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Compute $E[X^2]$. 
Expectation of functions of random variables (contd.)

Let $Y = X^2$. Observe that $Y$ is also a random variable. What are the values that $Y$ can take? 0, 1 and 4.

Let us compute the pmf of $Y$.

Note that, $P\{Y = 0\} = P\{X^2 = 0\} = P\{X = 0\} = 0.3$.

Similarly, $P\{Y = 1\} = P\{X^2 = 1\} = P\{X = 1\} = 0.5$.

$P\{Y = 4\} = P\{X^2 = 4\} = P\{X = 2\} = 0.2$.

Accordingly, $E[Y] = E[X^2] = 0 \cdot 0.3 + 1 \cdot 0.5 + 4 \cdot 0.2 = 1.3$. 
Expectation of functions of random variables (contd.)

Solution

Let $Y = X^2$. 

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Let \( Y = X^2 \). Observe that \( Y \) is also a random variable. What are the values that \( Y \) can take? 0, 1 and 4. Let us compute the pmf of \( Y \). Note that,
Expectation of functions of random variables (contd.)

Solution

Let $Y = X^2$. Observe that $Y$ is also a random variable. What are the values that $Y$ can take? 0, 1 and 4. Let us compute the pmf of $Y$. Note that,

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Expected values of functions of random variables (contd.)

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Accordingly,

$$E[Y] = E[X^2] = 0 \cdot 0.3 + 1 \cdot 0.5 + 4 \cdot 0.2 = 1.3$$
Expectation of functions - The Direct Approach

Theorem
If $X$ is a random variable with pmf $p(x)$, and $g(x)$ is any real-valued function, then,
$$E[g(X)] = \sum_{x:p(x) > 0} g(x) \cdot p(x).$$

Note
Applying the above theorem to the previous problem,
$$E[X^2] = 0^2 \cdot 0.3 + 1^2 \cdot 0.5 + 2^2 \cdot 0.2 = 1.3.$$
Expectation of functions - The Direct Approach

Theorem

If $X$ is a random variable with pmf $p()$, and $g()$ is any real-valued function, then,

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Expectation of functions - The Direct Approach

**Theorem**

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**Note**

Applying the above theorem to the previous problem,
Expectation of functions - The Direct Approach

**Theorem**

*If X is a random variable with pmf p(), and g() is any real-valued function, then,*

\[ E[g(X)] = \sum_{x: p(x) > 0} g(x) \cdot p(x) \]

**Note**

*Applying the above theorem to the previous problem,*

\[ E[X^2] = \]
Theorem

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Applying the above theorem to the previous problem,

$$E[X^2] = 0^2 \cdot 0.3 + 1^2 \cdot 0.5 + 2^2 \cdot 0.2 = 1.3$$
Outline

1 Preliminaries
   - Sample Space and Events
   - Defining Probabilities on Events
   - Conditional Probability
   - Independent Events
   - Bayes’ Formula

2 Random Variables
   - Expectation
   - Expectation of a function of a random variable
   - Linearity of Expectation
   - Variance

3 Identities

4 Variance of some common random variables
Let $X_1, X_2, \ldots, X_n$ denote $n$ random variables, defined over some probability space. Let $a_1, a_2, \ldots, a_n$ denote $n$ constants. Then,

$$E\left[\sum_{i=1}^{n} a_i \cdot X_i\right] = \sum_{i=1}^{n} a_i \cdot E[X_i]$$

Note that linearity of expectation holds even when the random variables are not independent.

Example: What is the expected value of the sum of the upturned faces, when two fair dice are tossed?
Linearity of Expectation

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Another Application

Example

Compute the expected value of the Binomial random variable.

Solution

Define

\[ X_i = \begin{cases} 1, & \text{if the } i\text{th trial is a success} \\ 0, & \text{otherwise} \end{cases} \]

Accordingly, the Binomial random variable (say \( X \)) can be expressed as:

\[ X = X_1 + X_2 + \ldots + X_n \]

However, each \( X_i \) is Bernoulli random variable with probability of success \( p \).

Hence, using linearity of expectation,

\[
E[X] = E[X_1 + X_2 + \ldots + X_n] = \sum_{i=1}^{n} E[X_i] = n \cdot p
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Definition

Let $X$ denote a random variable. The variance of $X$, denoted by $\text{Var}[X]$, is defined as:

$$\text{Var}[X] = E[(X - E[X])^2].$$

Note

$\text{Var}[X] = E[X^2] - (E[X])^2$. 

Subramani Probability Theory
Variance

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Partial linearity of Variance
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Partial linearity of Variance

1. \( \text{Var}(\sum_{i=1}^{n} (X_i)) = \sum_{i=1}^{n} \text{Var}(X_i) \), if \( X_1, X_2, \ldots, X_n \) are independent random variables.
Bernoulli Variable

Computation

For some $p$, $0 \leq p \leq 1$,

$$p(0) = (1 - p)$$

$$p(1) = p$$

$$E[X] = 0 \cdot (1 - p) + 1 \cdot p = p$$

$$E[X^2] = 1 \cdot p + 0 \cdot (1 - p) = p$$

$$\text{Var}[X] = E[X^2] - (E[X])^2 = p - p^2 = p \cdot (1 - p)$$
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\]
Observe that if $X$ is a binomially distributed random variable with parameters $n$ and $p$, then it can be expressed as a sum of $n$ independent Bernoulli variables, i.e., $X = \sum_{i=1}^{n} X_i$, where each $X_i$ is a Bernoulli random variable with parameter $p$.

Using the identity on the variance of a sum, we conclude that $\text{Var}(X) = n \cdot p \cdot (1 - p)$. 

Binomial Variable
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Geometric Variable

Statement

If $X$ is a geometric random variable with parameter $p$, 

$$\text{Var}(X) = \frac{1 - p}{p^2}$$
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