Dynamic programming and Finance Applications

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Applicability of decision Models to Finance

- In finance, the relationships between the variables are usually well defined, lending the problem to empirical analysis.
- For example, the way in which an increase in the proportion of a portfolio invested in a particular asset affects the mean and variance of the portfolio is clear.

Dynamic Programming

- Incremental decision making lends itself to dynamic programming approach.
- In dynamic programming, the optimal solution for a problem is obtained by assembling optimal solutions for sub-problems.
- For example, in portfolio formation, addition of the next asset to the portfolio is dependent on the existing portfolio.

Defining Basics

- Option contract
- A contract that allows you to
- buy (call) or sell(put)
- an asset (e.g. Stock)
- at a fixed (exercise) price
- At (European) or by (American)
- a certain date

European Option

- Can be exercised only at maturity
- Value depends on
- Current Price of the asset
- Exercise price of the option
- Volatility of returns on the underlying asset
- Risk free rate
- Time to maturity
- Closed form solution available

American Option

- Can be exercised at any time prior to or at maturity.
- Possibility of early exercise makes valuation complicated.
- At each step decision to continue holding or exercise depends on the offered payout.
- No closed form solution
- Dynamic programming suitable

Using a binomial lattice to price a simple American option

- We are building a graph, whose nodes are labeled by a backward process.
- In comparison to the shortest path problem in pricing an American option the up or down movement in the graph is random and beyond your control; the only decision you can take is exercising the option or not.

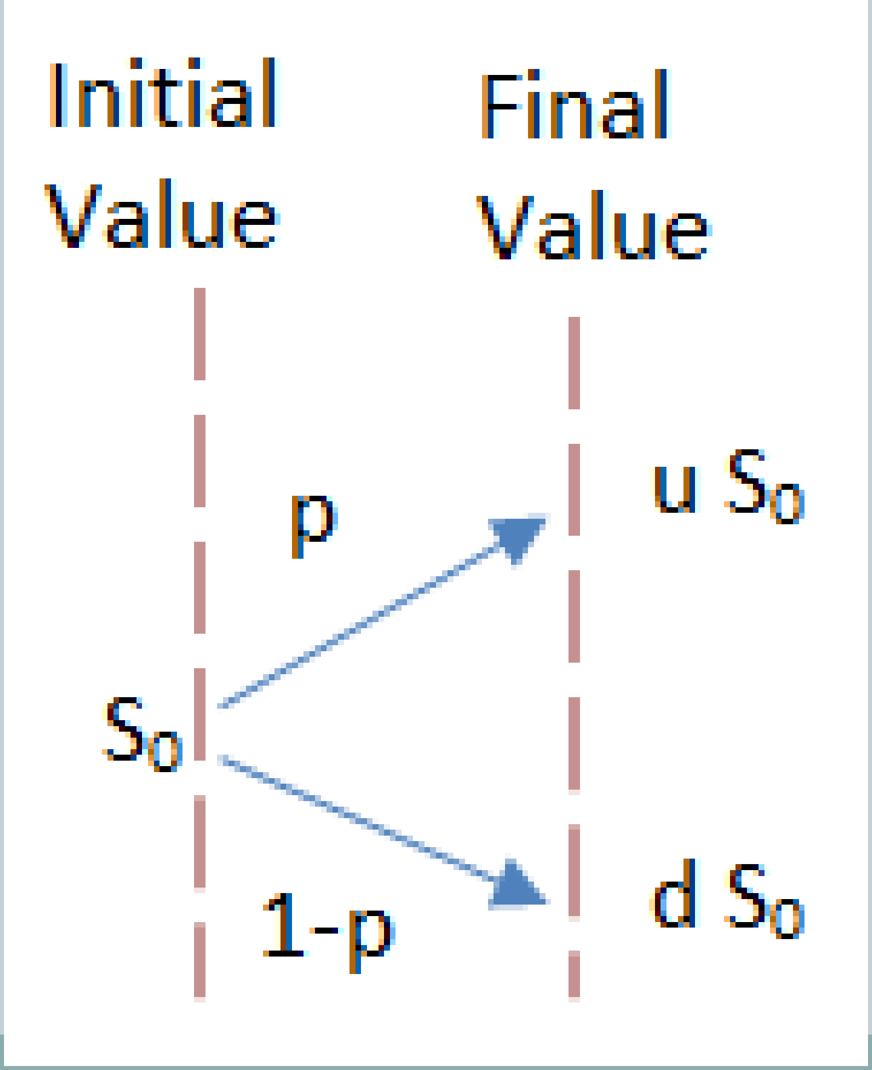
Binomial Option Pricing

- Based on a no-arbitrage assumption, mathematically simple, but powerful method to price options. Rather than relying on the solution to stochastic differential equations (which is often complex to implement), binomial option pricing is relatively simple to implement in Excel and is easily understood.
- No-arbitrage means that markets are efficient, and investments earn the risk-free rate of return.

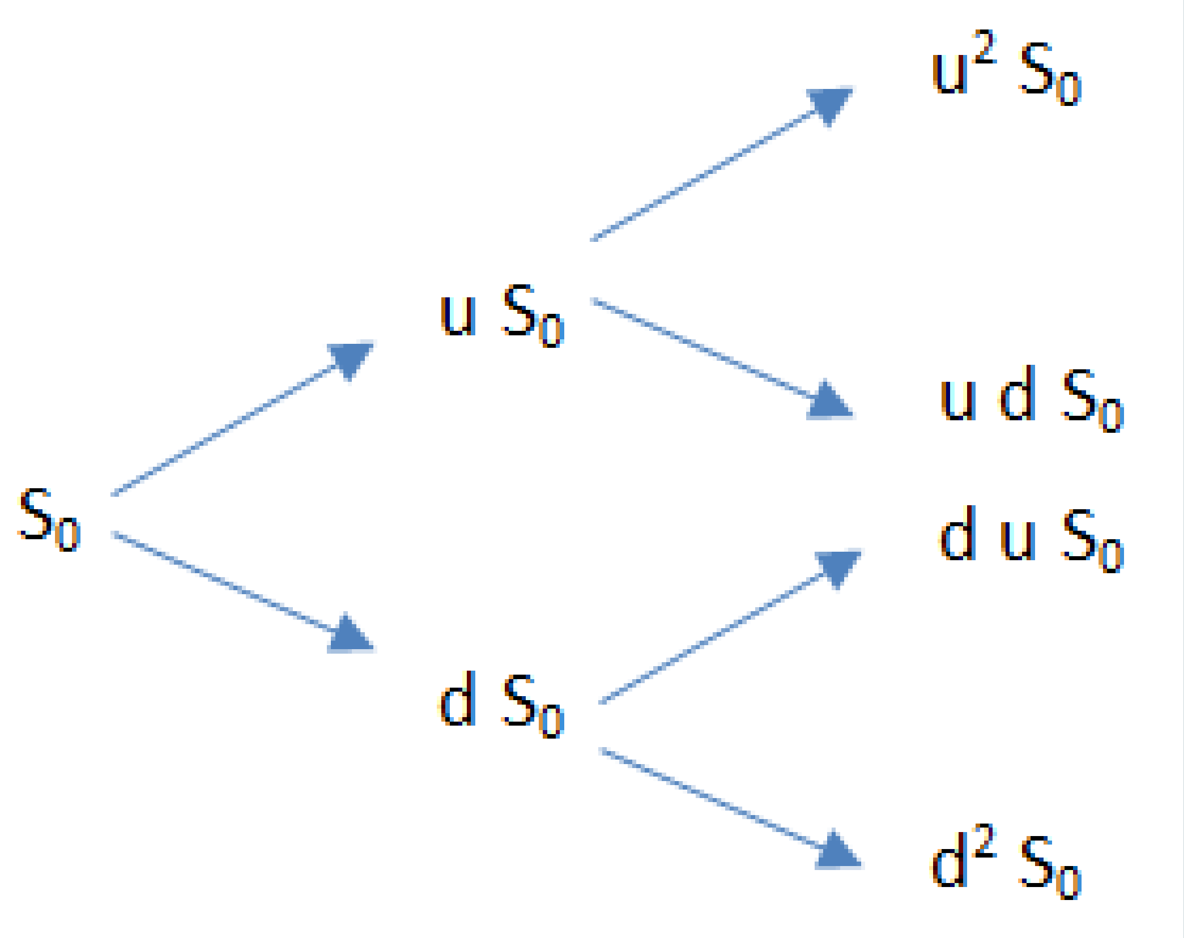
Initial layout of the problem

- Consider a stock (with an initial price of S_0) undergoing a random walk. Over a time step Δt, the stock has a probability p of rising by a factor u, and a probability 1-p of falling in price by a factor d.

Step one



Two Step Model



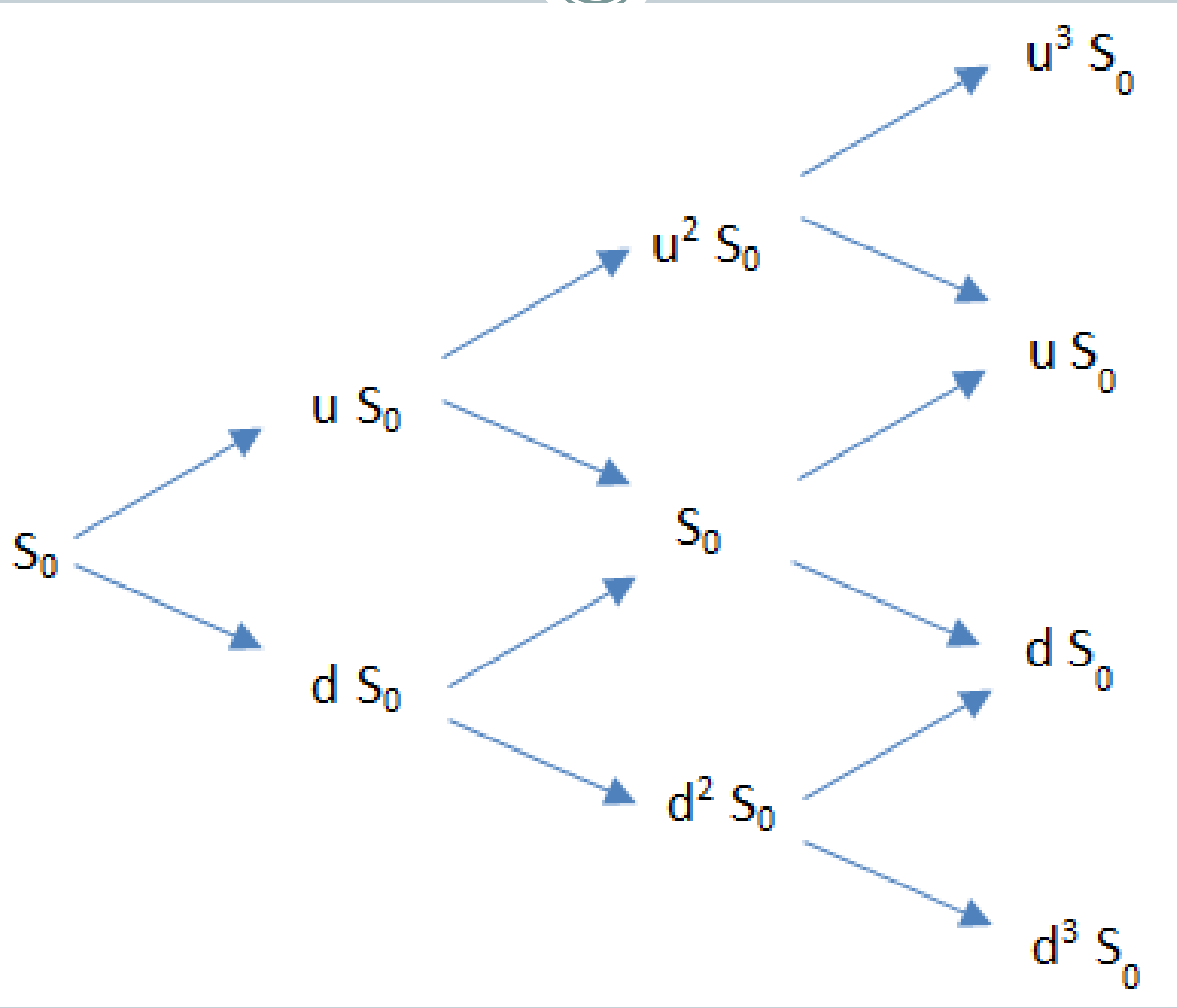
Cox, Ross and Rubenstein Model

$$p\,u + (1-p)\,d = e^{r\,\Delta t}$$

$$p\,u^2 + (1-p)d^2 - (e^{r\Delta t})^2 = \sigma^2 \Delta t$$

$$u = \frac{1}{d}$$

Recombinant Multi Step Model



Cox, Ross and Rubenstein Model

$$p = \frac{e^{r\,\Delta t} - d}{u - d}$$

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = e^{-\sigma\sqrt{\Delta t}}$$

Payoffs for Option Pricing

- V_N is the option price at the expiry node N, X is the strike or exercise price, S_N is the stock price at the expiry node N.
- Put: $V_N = \max(X - S_N, 0)$
- Call: $V_N = \max(S_N - X, 0)$

European options

- European Put: $V_n = e^{-r\Delta t}(p\,V_u + (1 - p)\,V_d)$
- European Call: $V_n = e^{-r\Delta t}(p\,V_u + (1 - p)\,V_d$

American Options

- American Put:
- $V_n = \max(X - S_n, e^{-r\Delta t}(p V_u + (1 - p) V_d))$
- American Call:
- $V_n = \max(S_n - X, e^{-r\Delta t}(p V_u + (1 - p) V_d))$
- $n \leq N$

Inputs for Binomial CRR

Parameters	
Stock Price S_0	4
Exercise Price X	4
Interest Rate r	0.04
Volatility	0.1
Time to Maturity	5
Number of Steps	5
Dividend Yield	0

Results		
	Binomial	BS Analytical
Call	0.807778	0.806877174
Put	0.082701	0.081800186

CRR Solution

	A	B	C	D	E	F	G
14							
15	**Calculations**						
16	Time Interval	1					
17	Up movement	1.10517	u				
18	Down movement	0.90484	d = 1/u				
19	Up probability	0.67873					
20	Discount Factor	0.96079					
21							
22	Step	0	1	2	3	4	5
23	Time	0	1	2	3	4	5
24							
25	Stock Price	4	4.420684	4.886	5.399435	5.967298791	6.594885083
26			3.61935	4	4.420684	4.885611033	5.39943523
27				3.275	3.61935	4	4.420683672
28					2.963273	3.274923012	3.619349672
29						2.681280184	2.963272883
30							2.426122639
31							
32	Option Price	0.80778	1.05072	1.349	1.70697	2.124141034	2.594885083
33			0.397128	0.554	0.764485	1.042453276	1.39943523
34				0.117	0.178901	0.27433683	0.420683672
35					0	0	0
36						0	0
37							0

$S_n u$

$d S_0$

$d^2 S_0$

S_N

Call Price

max($S_N - X$, 0)

Complex Assets

- Dynamic Programming can be used to price complex assets.
- Mortgage Backed securities
- Pricing a mortgage
- Present value relationship
- Complicated by early exercise of payoff option

Mortgage backed securities

- Mortgage backed securities (MBS) are created by the securitization of a pool of mortgages. For any specific mortgage, the borrower
- May repay the loan early - the prepayment option,
- or May default on the payments of capital and interest.
- These risks feed through to the owners of MBS.

Hybrid Securities

- MBS are hybrid securities, as they are variable interest rate securities with an early exercise option.
- MBS can be packaged in to Collateralized Mortgage Obligations (CMOs)
- Cash Flow from the MBS is separated in to principal only and interest only streams.

CMO Tranches

- Priority Tranches
- Principal Tranches
- Interest only Tranches
- Prepayment creates early payout gains for Principal Tranches
- Prepayment creates loss for Interest only Tranches

Option methods for CMOs

- Tranches can be priced as American Options
- Possibility of early exercise by borrowers.
- If interest rates fall borrower exercises the option to repay/refinance.
- Impact on P/I tranches significantly different.
- Repayment likelihood increases value for P tranches and reduces value for I tranches
- Default likelihood reduces value for P tranches and reduces value for I tranches

Risk and Return in CMOs

- P tranches are inherently less risky and therefore offer lower returns
- I tranches are more risky and offer higher returns.
- Pricing exercise for P and I tranches is complex but rewarding.