Quadratic Programming: Applications

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Outline

1. Mean-Variance Optimization

2. Brief mention of other MVO models

3. Maximizing the Sharpe Ratio

4. More Topics not covered

5. References

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Optimization Methods in Finance
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Markowitz’ theory of mean-variance optimization

Markowitz’ theory of mean-variance optimization (MVO) provides a mechanism for the selection of portfolios of securities (or asset classes) by considering the trade-off between risk and return. Consider assets $S_1, S_2, \ldots, S_n$ ($n \geq 2$) with random returns. Let $\mu_i$ and $\sigma_i$ denote the expected return and the standard deviation of the return of asset $S_i$. For $i \neq j$, $\rho_{ij}$ denotes the correlation coefficient of the returns of assets $S_i$ and $S_j$. Let $\mu = [\mu_1, \ldots, \mu_n]^T$ and $\Sigma = (\sigma_{ij})$ be the $n \times n$ symmetric covariance matrix with $\sigma_{ii} = \sigma_i^2$ and $\sigma_{ij} = \rho_{ij} \cdot \sigma_i \cdot \sigma_j$ for $i \neq j$. 

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Expected return and variance of the portfolio

If we let $x_i$ denote the proportion of the total funds invested in $S_i$, then the expected return and variance of the portfolio $x = (x_1, \cdots, x_n)$ can be represented as follows:

$$E[x] = \mu_1 \cdot x_1 + \cdots + \mu_n \cdot x_n = \mu^T \cdot x,$$

and

$$\text{Var}[x] = \sum_{i,j} \rho_{ij} \cdot \sigma_i \cdot \sigma_j \cdot x_i \cdot x_j = x^T \cdot \Sigma \cdot x,$$

where $\rho_{ii} \equiv 1$. Since variance is always nonnegative, it follows that $x^T \cdot \Sigma \cdot x \geq 0$ for any $x$, i.e., $\Sigma$ is positive semidefinite.
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Since variance is always nonnegative, it follows that $\mathbf{x}^T \cdot \Sigma \cdot \mathbf{x} \geq 0$ for any $\mathbf{x}$, i.e., $\Sigma$ is positive semidefinite.
Assumptions and constraints

We will assume that $\Sigma$ is positive definite. This is essentially equivalent to assuming that there are no redundant assets in our collection $S_1, S_2, \cdots, S_n$. We also assume that the set of admissible portfolios is a nonempty polyhedral set and represent it as

$$X := \{ x : A \cdot x = b, C \cdot x \geq d \},$$

where $A$ is an $m \times n$ matrix, $b$ is an $m$-dimensional vector, $C$ is a $p \times n$ matrix, and $d$ is a $p$-dimensional vector.

In particular, one of the constraints in the set $X$ is:

$$\sum_{i=1}^{n} x_i = 1.$$
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Markowitz theory of mean-variance optimization

The collection of efficient portfolios form the efficient frontier of the portfolio universe.

The efficient frontier is often represented as a curve in a two-dimensional graph where the coordinates of a plotted point corresponds to the standard deviation and the expected return of an efficient portfolio.

When we assume that $\Sigma$ is positive definite, the variance is a strictly convex function of the portfolio variables and there exists a unique portfolio in $\mathcal{X}$ that has the minimum variance. Let us denote this portfolio with $x_{\text{min}}$ and its return $\mu^T \cdot x_{\text{min}}$ with $R_{\text{min}}$.

(Note that $x_{\text{min}}$ is an efficient portfolio.) We let $R_{\text{max}}$ denote the maximum return for an admissible portfolio.
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An efficient portfolio is a portfolio with the maximal expected return among all portfolios with the same variance, or alternatively, a portfolio with the minimum variance among all portfolios that have at least a certain expected return. The collection of efficient portfolios forms the efficient frontier of the portfolio universe.

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Markowitz theory of mean-variance optimization
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Markowitz MVO problem formulation

Find the minimum variance portfolio that yields at least a target value of expected return. Mathematically, this formulation produces a quadratic programming problem:

\[
\min \quad x^T \Sigma x \quad \text{subject to} \quad \mu^T x \geq R \quad \text{and} \quad C x \geq d
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By solving this problem for values of \( R \) ranging between \( R_{\text{min}} \) and \( R_{\text{max}} \), we obtain all efficient portfolios.
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We have a convex QP problem for which the first-order conditions are both necessary and sufficient for optimality. Thus, \( x \in \mathbb{R} \) is an optimal solution of the problem if and only if there exists \( \lambda \in \mathbb{R} \), \( \gamma \in \mathbb{R}^m \), and \( \gamma \in \mathbb{R}^p \) satisfying the following KKT conditions:

\[
\begin{align*}
\Sigma \cdot x - \lambda \cdot \mu - A^T \cdot \gamma_E - C^T \cdot \gamma_I &= 0 \\
\mu^T \cdot x &\geq R \\
A \cdot x &= b \\
C \cdot x &\geq d \\
\lambda &\geq 0 \\
\gamma_I &\geq 0 \\
\gamma_I^T \cdot (C \cdot x - d) &= 0
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We have a convex QP problem for which the first-order conditions are both necessary and sufficient for optimality. Thus, $\mathbf{x} \in \mathbb{R}^n$ is an optimal solution of the problem if and only if there exist $\lambda \in \mathbb{R}$, $\gamma \in \mathbb{R}^m$, and $\gamma \in \mathbb{R}^p$ satisfying the following KKT conditions:

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\begin{align*}
\mathbf{\Sigma} \cdot \mathbf{x} - \lambda \cdot \mathbf{\mu} - \mathbf{A}^T \cdot \gamma - \mathbf{C}^T \cdot \gamma &= 0, \\
\mathbf{\mu}^T \cdot \mathbf{x} &\geq R, \\
\mathbf{A} \cdot \mathbf{x} &= b, \\
\mathbf{C} \cdot \mathbf{x} &\geq d, \\
\lambda &\geq 0, \\
\lambda \cdot (\mathbf{\mu}^T \cdot \mathbf{x} - R) &= 0, \\
\gamma \cdot \mathbf{I} &= 0, \\
\gamma \cdot \mathbf{C} \cdot \mathbf{x} - d &= 0.
\end{align*}
$$
Markowitz theory of mean-variance optimization

KKT conditions

- We have a convex QP problem for which the first-order conditions are both necessary and sufficient for optimality.
Markowitz theory of mean-variance optimization

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$$
\Sigma \cdot \mathbf{x}_R - \lambda_R \cdot \mu - \mathbf{A}^T \cdot \gamma_E - \mathbf{C}^T \cdot \gamma_I = \mathbf{0}
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$$
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Markowitz theory of mean-variance optimization

Example
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Z. Donovan and M. Xu
Optimization Methods in Finance
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Markowitz theory of mean-variance optimization

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<td>5588.19</td>
<td>1366.73</td>
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</table>

Z. Donovan and M. Xu  Optimization Methods in Finance
Calculating rates of return

Let $I_{it}$ denote the "total return" for asset $i = 1, 2, 3$ and $t = 0, \cdots, T$, where $t = 0$ corresponds to 1960 and $t = T$ corresponds to 2003. For each asset $i$, we can convert the raw data $I_{it}$, $t = 0, \cdots, T$, given in the previous table into rates of return $r_{it}$, $t = 1, \cdots, T$, using the formula

$$r_{it} = \frac{I_{i,t} - I_{i,t-1}}{I_{i,t-1}}$$

These rates of returns are shown in the next table.
Calculating rates of return
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Markowitz theory of mean-variance optimization

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Markowitz theory of mean-variance optimization

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Markowitz theory of mean-variance optimization

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<tr>
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Markowitz theory of mean-variance optimization

Table: Rates of return for stocks, bonds, and money market

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Markowitz theory of mean-variance optimization

Let $R_i$ denote the random rate of return of asset $i$. From the historical data, we can compute the arithmetic mean rate of return for each asset:

$\bar{r}_i = \frac{1}{T} \sum_{t=1}^{T} r_{it}$

This gives:

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<tr>
<th>Stocks</th>
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</tr>
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<tbody>
<tr>
<td>12.06%</td>
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Z. Donovan and M. Xu

Optimization Methods in Finance
Markowitz theory of mean-variance optimization

**Arithmetic mean**

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Academic content here

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<tbody>
<tr>
<td>Arithmetic mean $\bar{r}_i$</td>
<td>12.06%</td>
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Markowitz theory of mean-variance optimization

Geometric mean

Since the rates of return are multiplicative over time, we prefer to use the geometric mean instead of the arithmetic mean. The geometric mean is the constant yearly rate of return that needs to be applied in years $t = 0, \ldots, (T-1)$ in order to get the compounded total return $I_T^T$, starting from $I_i$. The formula for the geometric mean is:

$$\mu_i = \left( \prod_{t=1}^{T} (1 + r_{it}) \right)^{1/(T-1)}$$

We get the following results:

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Z. Donovan and M. Xu

Optimization Methods in Finance
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Markowitz theory of mean-variance optimization
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Markowitz theory of mean-variance optimization

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Markowitz theory of mean-variance optimization

We also compute the covariance matrix:

\[ \text{cov}(R_i, R_j) = \sum_{t=1}^{T} \left( r_{it} - \bar{r}_i \right) \left( r_{jt} - \bar{r}_j \right) \]

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<tr>
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<td>0.00021</td>
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<tr>
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Markowitz theory of mean-variance optimization

Covariance matrix

\[
\text{cov}(R_i, R_j) = \frac{1}{T} \sum_{i=1}^{T} (r_{it} - \bar{r}_i)(r_{jt} - \bar{r}_j)
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</tr>
<tr>
<td>Bonds</td>
<td>0.0039</td>
<td>0.0111</td>
</tr>
<tr>
<td>MM</td>
<td>0.0002</td>
<td>-0.0002</td>
</tr>
</tbody>
</table>

Z. Donovan and M. Xu

Optimization Methods in Finance
Markowitz theory of mean-variance optimization

**Covariance matrix**

- We also compute the covariance matrix:

<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>Stocks</td>
<td>0.02778</td>
<td>0.00387</td>
<td>0.00021</td>
</tr>
<tr>
<td>Bonds</td>
<td>0.00387</td>
<td>0.01112</td>
<td>-0.00020</td>
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<tr>
<td>MM</td>
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</tr>
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Covariance matrix

We also compute the covariance matrix:

\[ \text{cov}(R_i, R_j) = \frac{1}{T} \sum_{i=1}^{T} (r_{it} - \bar{r}_i)(r_{jt} - \bar{r}_j) \]
Markowitz theory of mean-variance optimization

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The volatility (or standard deviation) of the rate of return on each asset is:

\[ \sigma_i = \sqrt{\text{cov}(R_i, R_i)} \]

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**Correlation Matrix**

\[
\begin{array}{ccc}
\text{Correlation} & \text{Stocks} & \text{Bonds} & \text{MM} \\
\text{Stocks} & 1 & 0.2199 & 0.0366 \\
\text{Bonds} & 0.2199 & 1 & -0.0545 \\
\text{MM} & 0.0366 & -0.0545 & 1 \\
\end{array}
\]

\[\rho_{ij} = \frac{\text{cov}(R_i, R_j)}{\sigma_i \cdot \sigma_j}\]

This gives:

- Correlation between Stocks and Bonds: 0.2199
- Correlation between Stocks and MM: 0.0366
- Correlation between Bonds and MM: -0.0545
Markowitz theory of mean-variance optimization

Correlation matrix

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Z. Donovan and M. Xu
Optimization Methods in Finance
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The quadratic program for this problem is as follows:

\[
\begin{align*}
\min & \quad \frac{1}{2} x^T \Sigma x + \mu^T x \\
\text{s.t.} & \quad R x_S + x_B + x_M = 1 \\
& \quad x_S, x_B, x_M \geq 0
\end{align*}
\]

Solving for $R = 6.5\%$ to $R = 10.5\%$ with increments of 0.5%, gives us the optimal portfolios shown in the next table.

Z. Donovan and M. Xu
Optimization Methods in Finance
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Markowitz theory of mean-variance optimization
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Table of efficient portfolios

<table>
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<tr>
<th>Rate of return R</th>
<th>Variance</th>
<th>Stocks</th>
<th>Bonds</th>
<th>RR</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.065</td>
<td>0.0010</td>
<td>0.03</td>
<td>0.10</td>
<td>0.87</td>
</tr>
<tr>
<td>0.070</td>
<td>0.0014</td>
<td>0.13</td>
<td>0.12</td>
<td>0.75</td>
</tr>
<tr>
<td>0.075</td>
<td>0.0026</td>
<td>0.24</td>
<td>0.14</td>
<td>0.62</td>
</tr>
<tr>
<td>0.080</td>
<td>0.0044</td>
<td>0.35</td>
<td>0.16</td>
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Markowitz theory of mean-variance optimization

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Example

Let us consider a portfolio of stocks constructed from a set of \( n \) stocks with known expected returns and covariance matrix, where \( n \) may be in the hundreds or thousands.
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Let us consider a portfolio of stocks constructed from a set of $n$ stocks with known expected returns and covariance matrix, where $n$ may be in the hundreds or thousands.

Z. Donovan and M. Xu Optimization Methods in Finance
Large-scale portfolio optimization

Issues with large-scale portfolios

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- Let us consider a portfolio of stocks constructed from a set of $n$ stocks with known expected returns and covariance matrix, where $n$ may be in the hundreds or thousands.
In general, there is no reason to expect that solutions to the Markowitz model will be well diversified portfolios. This model tends to produce portfolios with unreasonably large weights in certain asset classes. This is often attributed to estimation errors. Estimates that may be slightly “off” may lead the optimizer to chase phantom low-risk high-return opportunities by taking large positions. Positions chosen by this quadratic program may be subject to idiosyncratic risk (i.e., risk specific to an asset or small group of assets having little or no correlation with market risk). Practitioners often use additional constraints on the $x_i$'s to insulate themselves against estimation and model errors, and to ensure that the chosen portfolio is well diversified. For example, a limit $m$ may be imposed on the size of each $x_i$, say $x_i \leq m$ for $i = 1, \ldots, n$. 
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Practitioners often use additional constraints on the $x_i$s to insure themselves against estimation and model errors, and to ensure that the chosen portfolio is well diversified.

- For example, a limit $m$ may be imposed on the size of each $x_i$, say $x_i \leq m$ for $i = 1, \ldots, n$. 
One can also reduce sector risk by grouping together investments in securities of a sector and setting a limit on the exposure of this sector. For example, if $m_k$ is the maximum that can be invested in sector $k$, we add the constraint

$$\sum_{i \in \text{sector } k} x_i \leq m_k$$

Note that the more constraints one adds to a model, the more the objective value deteriorates. So, this approach to producing diversification can be quite costly.
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- Note that the more constraints one adds to a model, the more the objective value deteriorates.

- So, this approach to producing diversification can be quite costly.
Large-scale portfolio optimization
Large-scale portfolio optimization

Transaction costs

We can add a portfolio turnover constraint to ensure that the change between the current holdings $x_0$ and the desired portfolio $x$ is bounded by $h$.

To avoid big changes when reoptimizing the portfolio, turnover constraints may be imposed.

Let $y_i$ be the amount of asset $i$ bought and $z_i$ the amount sold.

We write

$$x_i - x_0 i \leq y_i, \quad y_i \geq 0$$

$$x_0 i - x_i \leq z_i, \quad z_i \geq 0$$

$$\sum_{i=1}^{n} (y_i + z_i) \leq h$$

Z. Donovan and M. Xu

Optimization Methods in Finance
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Large-scale portfolio optimization

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Large-scale portfolio optimization

\[ \min \frac{1}{2} \sum_{i,j=1}^{n} \sigma_{ij} x_i x_j \]

subject to

\[ \frac{1}{n} \sum_{i=1}^{n} (\mu_i x_i - t_i y_i - t'_i z_i) \geq R \]

\[ x_i \leq y_i, \quad \text{for } i = 1, \ldots, n \]

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Z. Donovan and M. Xu

Optimization Methods in Finance
Large-scale portfolio optimization

The reoptimized portfolio is obtained by solving the following QP problem:

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\text{min} \quad \frac{1}{2} \cdot \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} \cdot x_i \cdot x_j
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The Markowitz model gives us an optimal portfolio assuming that we have perfect information on the $\mu_i$ and $\sigma_{ij}$ for the assets that we are considering. Therefore, an important practical issue is the estimation of the $\mu_i$ and $\sigma_{ij}$.

A reasonable approach for estimating these data is the use of time series of past returns ($r_{it} = \text{return of asset } i \text{ from time } t-1 \text{ to time } t$, where $i = 1, \ldots, n, t = 1, \ldots, T$).

Unfortunately, it has been observed that small changes in the time series $r_{it}$ lead to changes in the $\mu_i$ and $\sigma_{ij}$.

Such changes often lead to significant changes in the "optimal" portfolio. Markowitz recommends using $\beta$s (unknown regression parameters of the securities) to calculate the $\mu_i$ and $\sigma_{ij}$.

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The fundamental weakness of the Markowitz model remains, no matter how cleverly the $\mu_i$ and $\sigma_{ij}$ are computed. The solution is extremely sensitive to small changes in the data. Only one small change in $\mu_i$ may produce a totally different portfolio $x$.

So, what can be done in practice to overcome this problem, or at least reduce it? Michaud recommends resampling returns from historical data to generate alternative $\mu$ and $\sigma$ estimates, solving the MVO problem repeatedly with inputs generated this way, and then combining the optimal portfolios obtained in this manner. Robust optimization approaches provide an alternative strategy to mitigate the input sensitivity in MVO models. Another interesting approach is the Black-Litterman model, which allows investors to combine their unique views regarding the performance of various assets with the market equilibrium in a manner that results in intuitive, diversified portfolios.
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The Black-Litterman Model

The expected return vector $\mu$ is assumed to have a probability distribution that is the product of two multivariate normal distributions. The first distribution represents the returns at market equilibrium, with mean $\pi$ and covariance matrix $\tau \cdot \Sigma$, where $\tau$ is a small constant and $\Sigma = (\sigma_{ij})$ denotes the covariance matrix of asset returns.

Note that the factor $\tau$ should be small since the variance $\tau \cdot \sigma^2_i$ of the random variable $\mu_i$ is typically much smaller than the variance $\sigma^2_i$ of the underlying asset returns.

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Optimization Methods in Finance
The Black-Litterman Model

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The second distribution represents the investor's view about the $\mu_i$s. These views are expressed as $P \cdot \mu = q + \epsilon$, where $P$ is a $k \times n$ matrix, $q$ is a $k$-dimensional vector provided by the investor, $\epsilon$ is a normally distributed random vector with mean $0$. 

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The Black-Litterman Model

Black and Litterman use \( \bar{\mu} \) as the vector of expected returns in the Markowitz model. \( \Omega \) is the diagonal covariance matrix. The stronger the investor's view, the smaller the corresponding \( \omega_i = \Omega_{ii} \).
The Black-Litterman Model

The resulting distribution for $\mu$ is a multivariate normal distribution with mean $\bar{\mu} = \left[ (\tau \cdot \Sigma)^{-1} + P^T \cdot \Omega^{-1} \cdot P \right]^{-1} \cdot \left[ (\tau \cdot \Sigma)^{-1} \cdot \pi + P^T \cdot \Omega^{-1} \cdot q \right]$.

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Optimization Methods in Finance
The Black-Litterman Model

- The resulting distribution for $\mu$ is a multivariate normal distribution with mean

$$\bar{\mu} = (\tau \cdot \Sigma)^{-1} + P^T \cdot \Omega^{-1} \cdot P - \Omega^{-1} \cdot (\tau \cdot \Sigma)^{-1} \cdot \pi + \Omega^{-1} .$$

Black and Litterman use $\bar{\mu}$ as the vector of expected returns in the Markowitz model. $\Omega$ is the diagonal covariance matrix. The stronger the investor's view, the smaller the corresponding $\omega_i = \Omega_{ii}$. 

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Optimization Methods in Finance
The Black-Litterman Model

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- The resulting distribution for $\mu$ is a multivariate normal distribution with mean

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The resulting distribution for \( \mu \) is a multivariate normal distribution with mean

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The Black-Litterman Model

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The Black-Litterman Model

Example (Illustrating the Black-Litterman approach)

Using our previous MVO example, the expected returns on Stocks, Bonds, and Money Market were computed to be

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The Black-Litterman Model

We need to choose the value of the small constant $\tau$. So, take $\tau = 0.1$.

We have two views that we would like to incorporate into the model. First, we hold a strong view that the Money Market rate will be 2% next year. Second, we also hold the view that S&P 500 will outperform 10-year Treasury Bonds by 5%, but we are not as confident about this view.

These two views can be expressed as follows:

- $\mu^M = 0.02$ (strong view):
  - $\omega_1 = 0.00001$

- $\mu_{S - B} = 0.05$ (weaker view):
  - $\omega_2 = 0.001$

Thus, $P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$, $q = \begin{pmatrix} 0.02 \\ 0.05 \end{pmatrix}$ and $\Omega = \begin{pmatrix} 0.00001 & 0 \\ 0 & 0.001 \end{pmatrix}$. 

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$\mu_M = 0.02$ strong view: $\omega_1 = 0.00001$,

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The Black-Litterman Model

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- We need to choose the value of the small constant $\tau$. So, take $\tau = 0.1$. 

$$
\begin{align*}
\mu_M &= 0.02, \\
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\end{align*}
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\mathbf{P} &= \begin{pmatrix} 0 & 0.1 \\ 1 & -1 \end{pmatrix}, \\
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The Black-Litterman Model

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The Black-Litterman Model

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The Black-Litterman Model

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- First, we hold a strong view that the Money Market rate will be 2% next year.
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The Black-Litterman Model

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The Black-Litterman Model

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Mean-Variance Optimization
Brief mention of other MVO models
Maximizing the Sharpe Ratio
More Topics not covered
References

The Black-Litterman Model

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The Black-Litterman Model

Example (to illustrate the Black-Litterman approach)

We solve the same QP expect for the modified expected return constraint:

\[
\begin{align*}
\min 1 & \cdot \mathbf{x}^T \Sigma \mathbf{x} + 2 & \cdot \mathbf{0.02778} \cdot \mathbf{x}^T \mathbf{S} + 2 & \cdot \mathbf{0.00387} \cdot \mathbf{x}^T \mathbf{B} + \mathbf{0.00021} \cdot \mathbf{x}^T \mathbf{M} + \mathbf{0.01112} \\
& \geq R \cdot (\mathbf{x}_S + \mathbf{x}_B + \mathbf{x}_M) = 1 \\
& \mathbf{x}_S, \mathbf{x}_B, \mathbf{x}_M \geq 0
\end{align*}
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The Black-Litterman Model

Example (to illustrate the Black-Litterman approach)
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- Applying our formula to compute $\bar{\mu}$ gives:

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- We solve the same QP except for the modified expected return constraint:
The Black-Litterman Model

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\min \frac{1}{2} \left[ 0.02778 \cdot x_S^2 + 2 \cdot 0.00387 \cdot x_S \cdot x_B + 2 \cdot 0.00021 \cdot x_S \cdot x_M \\
+ 0.01112 \cdot x_B^2 - 2 \cdot 0.00020 \cdot x_B \cdot x_M + 0.00115 \cdot x_M^2 \right]
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$$\quad 0.1177 \cdot x_S + 0.0751 \cdot x_B + 0.0234 \cdot x_M \geq R$$
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$$x_S + x_B + x_M = 1$$

$$x_S, x_B, x_M \geq 0$$
The Black-Litterman Model

The Black-Litterman Model is a method for portfolio optimization that incorporates market views and subjective beliefs about expected returns into the mean-variance framework. It is an extension of the traditional mean-variance optimization (MVO) model, which was developed by Harry Markowitz.

The Black-Litterman model addresses some of the limitations of the traditional MVO model, such as the sensitivity to input errors and the difficulty in incorporating views from market participants into the optimization process.

Solving this QP for $R = 4\%$ to $R = 11.5\%$ with increments of 0.5% results in the optimal portfolios shown in the table below.

<table>
<thead>
<tr>
<th>Rate of return R</th>
<th>Variance</th>
<th>Stocks</th>
<th>Bonds</th>
<th>MM</th>
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<tbody>
<tr>
<td>0.040</td>
<td>0.0012</td>
<td>0.08</td>
<td>0.17</td>
<td>0.75</td>
</tr>
<tr>
<td>0.045</td>
<td>0.0015</td>
<td>0.11</td>
<td>0.21</td>
<td>0.68</td>
</tr>
<tr>
<td>0.050</td>
<td>0.0020</td>
<td>0.15</td>
<td>0.24</td>
<td>0.61</td>
</tr>
<tr>
<td>0.055</td>
<td>0.0025</td>
<td>0.18</td>
<td>0.28</td>
<td>0.54</td>
</tr>
<tr>
<td>0.060</td>
<td>0.0032</td>
<td>0.22</td>
<td>0.31</td>
<td>0.47</td>
</tr>
<tr>
<td>0.065</td>
<td>0.0039</td>
<td>0.25</td>
<td>0.35</td>
<td>0.40</td>
</tr>
<tr>
<td>0.070</td>
<td>0.0048</td>
<td>0.28</td>
<td>0.39</td>
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</tr>
<tr>
<td>0.075</td>
<td>0.0059</td>
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<td>0.26</td>
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<td>0.085</td>
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<td>0.0133</td>
<td>0.58</td>
<td>0.42</td>
<td>0.00</td>
</tr>
<tr>
<td>0.105</td>
<td>0.0163</td>
<td>0.70</td>
<td>0.30</td>
<td>0.00</td>
</tr>
<tr>
<td>0.110</td>
<td>0.0202</td>
<td>0.82</td>
<td>0.18</td>
<td>0.00</td>
</tr>
<tr>
<td>0.115</td>
<td>0.0249</td>
<td>0.94</td>
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Z. Donovan and M. Xu

Optimization Methods in Finance
### The Black-Litterman Model

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Black-Litterman efficient portfolios

- Solving this QP for $R = 4\%$ to $R = 11.5\%$ with increments of 0.5\% results in the optimal portfolios shown in the table below.

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The Sharpe Ratio

The Sharpe Ratio is a measure for calculating risk-adjusted return and this ratio has become the industry standard for such calculations. The Sharpe Ratio (reward-to-volatility ratio) is the average return earned in excess of the risk-free rate per unit of volatility or total risk. It was firstly introduced by Nobel Laureate William F. Sharpe to measure the performance of mutual funds in 1966. Subtracting the risk-free rate from the mean return, the performance associated with risk-taking activities can be isolated. One intuition of this calculation is that a portfolio engaging in “zero risk” investment, such as the purchase of U.S. Treasury bills (for which the expected return is the risk-free rate), has a Sharpe ratio of exactly zero. Generally, the greater the value of the Sharpe ratio, the more attractive the risk-adjusted return.
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Capital Allocation Line (CAL)
Capital Allocation Line (CAL)

Notation

\( r_f \) = rate of return on the risk-free asset

\( r_p \) = rate of return on the risky portfolio

\( r_C \) = rate of return on the complete portfolio (including both the risk-free asset and the risky portfolio)

\( y \) = proportion of the investment budget to be placed in the risky portfolio

\( \sigma_p \) = standard deviation of the return on the risky portfolio

\( \sigma_C \) = standard deviation of the return on the complete portfolio

Z. Donovan and M. Xu

Optimization Methods in Finance
Capital Allocation Line (CAL)

Notation

- $r_f = \text{rate of return on the risk-free asset}$
Capital Allocation Line (CAL)

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Capital Allocation Line (CAL)

Characterization of the Complete Portfolio

Rate of return $r_C = y \cdot r_p + (1 - y) \cdot r_f$

Expected rate of return $E(r_C) = y \cdot E(r_p) + (1 - y) \cdot r_f$

Variance $\sigma_C^2 = y^2 \cdot \sigma_p^2 + (1 - y)^2 \cdot \sigma_f^2 + 2 \cdot y \cdot (1 - y) \cdot \text{cov}(r_p, r_f)$

Standard deviation $\sigma_C = \sqrt{y^2 \cdot \sigma_p^2 + (1 - y)^2 \cdot \sigma_f^2 + 2 \cdot y \cdot (1 - y) \cdot \text{cov}(r_p, r_f)}$

More Topics not covered

References

Z. Donovan and M. Xu

Optimization Methods in Finance
Capital Allocation Line (CAL)

Characterization of the Complete Portfolio
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- Rate of return
Capital Allocation Line (CAL)

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- Variance
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- Standard deviation
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Capital Allocation Line (CAL)

Available Complete Portfolios

Solve for $y$

$y = \sigma_C / \sigma_p$

Replace in the equation for the expected rate of return $E(r_C) = r_f + \sigma_C \cdot \left[ E(r_p) - r_f \right]$

This defines a line in the mean-variance space – the capital allocation line (CAL)

Slope of CAL (Sharpe Ratio):

$\left[ E(r_p) - r_f \right] / \sigma_p$

or

$\left[ \mu_T \cdot x - r_f \right] / \left( x^T \cdot \Sigma \cdot x \right)^{1/2}$

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Optimization Methods in Finance
Capital Allocation Line (CAL)

Available Complete Portfolios
Capital Allocation Line (CAL)

Available Complete Portfolios

- Solve for $y$

This defines a line in the mean-variance space – the capital allocation line (CAL).

Slope of CAL (Sharpe Ratio):

$$\frac{E(r_p) - r_f}{\sigma_p}$$

or

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Capital Allocation Line (CAL)

Available Complete Portfolios

- Solve for $y$
  
  $$y = \frac{\sigma_C}{\sigma_p}$$
Capital Allocation Line (CAL)

Available Complete Portfolios

- Solve for \( y \)
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- Replace in the equation for the expected rate of return
Capital Allocation Line (CAL)

Available Complete Portfolios

- Solve for $y$
  \[ y = \frac{\sigma_C}{\sigma_P} \]
- Replace in the equation for the expected rate of return
  \[ E(r_C) = r_f + \frac{\sigma_C}{\sigma_P} \cdot [E(r_p) - r_f] \]
Capital Allocation Line (CAL)

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  $$E(r_C) = r_f + \frac{\sigma_C}{\sigma_p} \cdot [E(r_p) - r_f] = r_f + \sigma_C \cdot \frac{[E(r_p) - r_f]}{\sigma_p}$$
### Capital Allocation Line (CAL)

<table>
<thead>
<tr>
<th>Available Complete Portfolios</th>
</tr>
</thead>
<tbody>
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Capital Allocation Line (CAL)

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Capital Allocation Line (CAL)

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Capital Allocation Line (CAL)
Capital Allocation Line (CAL)

Capital Allocation Line

\[
\text{E}(r) - \sigma_f 
\]

\[
P \text{E}(r_p) - \sigma_p 
\]
Capital Allocation Line (CAL)

Capital Allocation Line

\[ E(r) \]

\[ \sigma \]
Capital Allocation Line (CAL)

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Capital Allocation Line (CAL)

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\[ \sigma_p \]

\[ P \]

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\[ \sigma \]

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Capital Allocation Line (CAL)

\[ E(r) \]

\[ rf \]

\[ \sigma \]

\[ P \]
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\[ E(r) \]

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\[ \sigma_p \]

\[ \sigma \]

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\[ \sigma \]
Capital Allocation Line (CAL)

\[ r_f = 7\% \]
\[ E(r_p) = 15\% \]
\[ \sigma_p = 22\% \]
\[ y = 0.75 \]
\[ E(r_C) = 0.75 \cdot 15\% + 0.25 \cdot 7\% = 13\% \]
\[ \sigma_C = y \cdot \sigma_p = 0.75 \cdot 22\% = 16.5\% \]

Slope of CAL:
\[ \frac{E(r_p) - r_f}{\sigma_p} = \frac{8\%}{22\%} = 0.36 \]

Z. Donovan and M. Xu
Optimization Methods in Finance
Capital Allocation Line (CAL)

Example

\[ r_f = 7\% \]

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Example
Capital Allocation Line (CAL)

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Capital Allocation Line (CAL)

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Recall that the Markowitz Efficient Frontier is the set of all portfolios of which expected returns reach the maximum given a certain level of risk. We denote with $R_{\min}$ and $R_{\max}$ the minimum and maximum expected returns for efficient portfolios.

Define the function

$$\sigma(R) : [R_{\min}, R_{\max}] \rightarrow \mathbb{R}, \sigma(R) := (x^T \cdot \Sigma \cdot x)^{1/2},$$

where $x_R$ denotes the unique solution of the MVO problem. Since we assumed that $\Sigma$ is positive definite, it is easy to show that the function $\sigma(R)$ is strictly convex in its domain.

We will assume that $r_f < R_{\min}$, which is natural since the portfolio $x_{\min}$ has a positive risk associated with it while the risk-free asset does not.
Efficient Frontier

Recall that the Markowitz Efficient Frontier is the set of all portfolios from which expected returns reach the maximum given a certain level of risk. We denote with $R_{\text{min}}$ and $R_{\text{max}}$ the minimum and maximum expected returns for efficient portfolios.

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Maximize the Sharpe Ratio

Remark

Since CAL goes through a feasible point, the optimal CAL goes through a point on the efficient frontier and never goes above a point on the efficient frontier.

Maximizing the Sharpe Ratio

$E(r) - \sigma$

Efficient frontier

$x_{min}$

$x_{max}$

Feasible point

$r_f$

CAL

Optimal CAL

Optimal risky portfolio
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\[ E(r) \]

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\[ E(r) \] vs. \[ \sigma \]
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\[ E(r) \]

\[ \sigma \]

Efficient frontier

\[ x_{\text{min}} \]

\[ x_{\text{max}} \]
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\[ E(r) \]

\[ \sigma \]

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\[ x_{\text{max}} \]

Feasible point

\[ x_{\text{min}} \]

\[ r_f \]
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Maximizing the Sharpe Ratio

\[
E(r) \quad \sigma
\]

Efficient frontier

CAL

\(x_{\text{min}}\)

\(x_{\text{max}}\)

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\(r_f\)

Optimal risky portfolio

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Optimization Methods in Finance
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\[ \sigma \]

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Feasible point

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\( x_{\text{min}} \)
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The Optimal Risky Portfolio

The portfolio that maximizes the Sharpe ratio is found by solving the following problem:

$$\max_x \mu^T x - r_f \left( x^T \Sigma x \right)^{1/2}$$

subject to:

$$A \cdot x = b$$
$$C \cdot x \geq d$$

Remark

Although it has a nice polyhedral feasible region, its objective function is somewhat complicated and possibly non-concave. So it is not a convex optimization problem.
The Optimal Risky Portfolio

Optimal Risky Portfolio

The portfolio that maximizes the Sharpe ratio is found by solving the following problem:

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The portfolio that maximizes the Sharpe ratio is found by solving the following problem:

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Z. Donovan and M. Xu
Optimization Methods in Finance
The Optimal Risky Portfolio

Equivalent Quadratic Programming

We describe a direct method to obtain the optimal risky portfolio by constructing an equivalent convex quadratic programming problem.

Assumptions

1. We assume that $\sum_{i} x_i = 1$ for any feasible portfolio $x$. This is a natural assumption since the $x_i$'s are the proportions of the portfolio in different asset classes.

2. We assume that there exists a feasible portfolio $\hat{x}$ with $\mu^T \cdot \hat{x} > r_f$. If all feasible portfolios have expected return bounded by the risk-free rate, there is no need to optimize, the risk-free investment dominates all others.
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The Optimal Risky Portfolio

Proposition

Given a set \( X \) of feasible portfolios with the properties that \( e^T \cdot x = 1, \forall x \in X \) and \( \exists \hat{x} \in X \) such that \( \mu^T \cdot \hat{x} > r_f \), the portfolio \( x^* \) with the maximum Sharpe ratio in this set can be found by solving the following problem

\[
\min_{y, \kappa} y^T \cdot \Sigma \cdot y \quad s.t. \quad (y, \kappa) \in X^+, \quad (\mu - r_f \cdot e)^T \cdot y = 1,
\]

where \( X^+ : = \{ x \in \mathbb{R}^n, \kappa \in \mathbb{R} | \kappa > 0, x\kappa \in X} \cup (0, 0) \).

If \((y, \kappa)\) is the solution of this problem, then \( x^* = y\kappa \).

Remark

This is a quadratic program and can be solved by IPMs.
The Optimal Risky Portfolio

Proposition

Given a set $X$ of feasible portfolios with the properties that $e^T \cdot x = 1, \forall x \in X$ and $\exists \hat{x} \in X$ such that $\mu^T \cdot \hat{x} > r_f$, the portfolio $x^*$ with the maximum Sharpe ratio in this set can be found by solving the following problem

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If $(y, \kappa)$ is the solution of this problem, then $x^* = y \kappa$.

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This is a quadratic program and can be solved by IPMs.
Proposition

Given a set $\mathcal{X}$ of feasible portfolios with the properties that $\mathbf{e}^T \cdot \mathbf{x} = 1$, $\forall \mathbf{x} \in \mathcal{X}$ and $\exists \hat{\mathbf{x}} \in \mathcal{X}$ such that $\mathbf{\mu}^T \cdot \hat{\mathbf{x}} > r_f$, the portfolio $\mathbf{x}^*$ with the maximum Sharpe ratio in this set can be found by solving the following problem:

$$\min_{y, \kappa} y^T \cdot \Sigma \cdot y \quad s.t. \quad (y, \kappa) \in \mathcal{X}^+, \quad (\mathbf{\mu} - r_f \cdot \mathbf{e})^T \cdot y = 1,$$

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The Optimal Risky Portfolio

Proposition

Given a set \( \mathcal{X} \) of feasible portfolios with the properties that \( \mathbf{e}^T \cdot \mathbf{x} = 1, \forall \mathbf{x} \in \mathcal{X} \) and \( \exists \hat{\mathbf{x}} \in \mathcal{X} \) such that \( \mathbf{\mu}^T \cdot \hat{\mathbf{x}} > r_f \), the portfolio \( \mathbf{x}^* \) with the maximum Sharpe ratio in this set can be found by solving the following problem:

\[
\min_{(\mathbf{y}, \kappa)} y^T \cdot \Sigma \cdot y \quad s.t. (\mathbf{y}, \kappa) \in \mathcal{X}^+, (\mathbf{\mu} - r_f \cdot \mathbf{e})^T \cdot \mathbf{y} = 1,
\]

where \( \mathcal{X}^+ = \{ \mathbf{x} \in \mathbb{R}^n, \kappa \in \mathbb{R} | \kappa > 0, \mathbf{x} \kappa \in \mathcal{X} \} \cup (0, 0) \).

Remark

This is a quadratic program and can be solved by IPMs.
Proposition

Given a set $\mathcal{X}$ of feasible portfolios with the properties that $e^T \cdot x = 1, \forall x \in \mathcal{X}$ and $\exists \hat{x} \in \mathcal{X}$ such that $\mu^T \cdot \hat{x} > r_f$, the portfolio $x^*$ with the maximum Sharpe ratio in this set can be found by solving the following problem:

$$\min y^T \cdot \Sigma \cdot y \text{ s.t. } (y, \kappa) \in \mathcal{X}^+, (\mu - r_f \cdot e)^T \cdot y = 1,$$

where $\mathcal{X}^+ = \{ x \in \mathbb{R}^n, \kappa \in \mathbb{R} | \kappa > 0, x \kappa \in \mathcal{X} \} \cup (0, 0)$.

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The Optimal Risky Portfolio

**Proposition**

Given a set $\mathcal{X}$ of feasible portfolios with the properties that $\mathbf{e}^T \cdot \mathbf{x} = 1$, $\forall \mathbf{x} \in \mathcal{X}$ and $\exists \hat{\mathbf{x}} \in \mathcal{X}$ such that $\mu^T \cdot \hat{\mathbf{x}} > r_f$, the portfolio $\mathbf{x}^*$ with the maximum Sharpe ratio in this set can be found by solving the following problem

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The Optimal Risky Portfolio

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where

$$\mathcal{X}^+ := \{x \in \mathbb{R}^n, \kappa \in \mathbb{R} \mid \kappa > 0, \frac{x}{\kappa} \in \mathcal{X}\} \cup (0, 0).$$

If $(y, \kappa)$ is the solution of this problem, then $x^* = \frac{y}{\kappa}$. 

Remark

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Remark

- This is a quadratic program and can be solved by IPMs.
Proof of Proposition

By our second assumption, it suffices to consider only those $x$ for which $(\mu - r_f \cdot e)^T x > 0$.

Let us make the following change of variables:

$$\kappa = \frac{1}{(\mu - r_f \cdot e)^T x}$$

$$y = \kappa \cdot x$$

Then,

$$\sqrt{x^T \Sigma x} = \kappa \cdot \sqrt{y^T \Sigma y}$$

and the objective function can be written as

$$\frac{1}{\sqrt{y^T \Sigma y}}$$

in terms of the new variables.
The Optimal Risky Portfolio

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By our second assumption, it suffices to consider only those $\mathbf{x}$ for which

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The Optimal Risky Portfolio

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The Optimal Risky Portfolio

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$$

$$
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Then, $\sqrt{x^T \cdot \Sigma \cdot x} = \frac{1}{\kappa} \cdot \sqrt{y^T \cdot \Sigma \cdot y}$ and the objective function can be written as $1/\sqrt{y^T \cdot \Sigma \cdot y}$ in terms of the new variables.
Mean-Variance Optimization
Brief mention of other MVO models
Maximizing the Sharpe Ratio
More Topics not covered
References

The Optimal Risky Portfolio

Proof of Proposition (cont'd.)

Note also that \((\mu - rf \cdot e)^T x > 0, x \in X \iff \kappa > 0, y \kappa \in X, \kappa = 1(\mu - rf \cdot e)^T x \iff (\mu - rf \cdot e)^T y = 1\).

Since \((\mu - rf \cdot e) y = 1\) rules out \((0, 0)\) as a solution, replacing \((\kappa > 0, y, \kappa) \in X\) with \((y, \kappa) \in X^+\) does not affect the solutions – it just makes the feasible set a closed set.
The Optimal Risky Portfolio

Proof of Proposition (cont’d.)

Note also that \((\mu - r_f \cdot e)^T \cdot x > 0, x \in X\) \(\iff\) \(\kappa > 0, y \kappa \in X\), and \(\kappa = 1 \cdot (\mu - r_f \cdot e)^T \cdot x \iff (\mu - r_f \cdot e)^T \cdot y = 1\). Since \((\mu - r_f \cdot e)^T \cdot y = 1\) rules out \((0, 0)\) as a solution, replacing \((y, \kappa) \in X\) with \((y, \kappa) \in X^+\) does not affect the solutions – it just makes the feasible set a closed set.
Proof of Proposition (cont’d.)

Note also that

\[
(\mu - r_f \cdot e) \cdot x > 0, \quad x \in X \iff \kappa > 0, \quad (y, \kappa) \in X
\]

and

\[
(\mu - r_f \cdot e) \cdot x \iff (\mu - r_f \cdot e) \cdot y = \frac{1}{\kappa}.
\]

Since

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(\mu - r_f \cdot e) \cdot y = 1
\]
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\[(\mu - r_f \cdot e)^T \cdot x > 0, \; x \in \mathcal{X} \iff \kappa > 0, \; \frac{y}{\kappa} \in \mathcal{X},\]
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The Optimal Risky Portfolio

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The Optimal Risky Portfolio

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The Optimal Risky Portfolio

Exercise

If $X = \{ x | A \cdot x \geq b, C \cdot x = d \}$, show that $X^+ = \{ (x, \kappa) | A \cdot x - b \cdot \kappa \geq 0, C \cdot x - d \cdot \kappa = 0, \kappa \geq 0 \}$. 

Z. Donovan and M. Xu

Optimization Methods in Finance
The Optimal Risky Portfolio

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If $X = \{x \mid A \cdot x \geq b, C \cdot x = d\}$, show that $X^+ = \{(x, \kappa) \mid A \cdot x - b \cdot \kappa \geq 0, C \cdot x - d \cdot \kappa = 0, \kappa \geq 0\}$. 

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\]
The Optimal Risky Portfolio

Exercise

Consider the previous MVO example. The covariance matrix is given as

\[
\text{Covariance} \\
\begin{array}{ccc}
\text{Stocks} & \text{Bonds} & \text{MM} \\
\text{Stocks} & 0.02778 & 0.00387 & 0.00021 \\
\text{Bonds} & 0.00387 & 0.01112 & -0.00020 \\
\text{MM} & 0.00021 & -0.00020 & 0.00115 \\
\end{array}
\]

And the geometric mean is given as

\[
\text{Geometric mean} \\
\begin{array}{c}
\mu_i \\
10.73\% & 7.37\% & 6.27\% \\
\end{array}
\]

Also, the matrix

\[
A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}
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and

\[
b = 1.
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Assume that the risk-free return rate is 3\%.

Find the program of optimal risky portfolio and the equivalent quadratic programming problem.
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Assume that the risk-free return rate is 3%. Find the program of optimal risky portfolio and the equivalent quadratic programming problem.
The Optimal Risky Portfolio

\[ \Sigma = \begin{bmatrix} 0 & 0.02778 & 0.00387 & 0.00021 \\ 0.02778 & 0.00387 & 0.01112 & -0.00020 \\ 0.00387 & 0.01112 & 0.00115 & 0.00021 \\ 0.00021 & -0.00020 & 0.00115 & 0.00021 \end{bmatrix} \]

\[ \mu = \begin{bmatrix} 0.1073 \\ 0.0737 \\ 0.0627 \end{bmatrix} \]

So the program of optimal risky portfolio is

\[ \max 0.1073 \cdot x_S + 0.0737 \cdot x_B + 0.0627 \cdot x_M - 0.03 (0.02778 \cdot x_S^2 + 2 \cdot 0.00387 \cdot x_S \cdot x_B + 2 \cdot 0.00021 \cdot x_S \cdot x_M + 0.01112 \cdot x_B^2 - 2 \cdot 0.00020 \cdot x_B \cdot x_M + 0.00115 \cdot x_M^2) ^{1/2} \]
The Optimal Risky Portfolio

Solution

\[
\begin{bmatrix}
0.02778 & 0 \\
0.00387 & 0.00021 \\
-0.00020 & -0.00020
\end{bmatrix}
\]

\[
\begin{bmatrix}
0.1073 \\
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So the program of optimal risky portfolio is

\[
\max 0.1073 \cdot x_S + 0.0737 \cdot x_B + 0.0627 \cdot x_M - 0.03 (0.02778 \cdot x_S^2 + 2 \cdot 0.00387 \cdot x_S \cdot x_B + 2 \cdot 0.00021 \cdot x_S \cdot x_M + 0.01112 \cdot x_B^2 - 2 \cdot 0.00020 \cdot x_B \cdot x_M + 0.00115 \cdot x_M^2)
\]

\[
\frac{1}{2} x_S + x_B + x_M = 1
\]
The Optimal Risky Portfolio

Solution

\[ \Sigma = \begin{bmatrix} 0.02778 & 0.00387 & 0.00021 \\ 0.00387 & 0.01112 & -0.00020 \\ 0.00021 & -0.00020 & 0.00115 \end{bmatrix} \]

\[ \mu = \begin{bmatrix} 0.1073 \\ 0.0737 \\ 0.0627 \end{bmatrix} \]
The Optimal Risky Portfolio

Solution

\[ \Sigma = \begin{bmatrix} 0.02778 & 0.00387 & 0.00021 \\ 0.00387 & 0.01112 & -0.00020 \\ 0.00021 & -0.00020 & 0.00115 \end{bmatrix} \]

\[ \mu = \begin{bmatrix} 0.1073 \\ 0.0737 \\ 0.0627 \end{bmatrix} \]
The Optimal Risky Portfolio

Solution

\[ \Sigma = \begin{bmatrix} 0.02778 & 0.00387 & 0.00021 \\ 0.00387 & 0.01112 & -0.00020 \\ 0.00021 & -0.00020 & 0.00115 \end{bmatrix} \]

\[ \mu = \begin{bmatrix} 0.1073 \\ 0.0737 \\ 0.0627 \end{bmatrix} \]

So the program of optimal risky portfolio is
The Optimal Risky Portfolio

Solution

\[ \Sigma = \begin{bmatrix} 0.02778 & 0.00387 & 0.00021 \\ 0.00387 & 0.01112 & -0.00020 \\ 0.00021 & -0.00020 & 0.00115 \end{bmatrix} \]

\[ \mu = \begin{bmatrix} 0.1073 \\ 0.0737 \\ 0.0627 \end{bmatrix} \]

So the program of optimal risky portfolio is

\[
\begin{align*}
\text{max} & \quad 0.1073 \cdot x_S + 0.0737 \cdot x_B + 0.0627 \cdot x_M - 0.03 \\
& \frac{0.02778 \cdot x_S^2 + 2 \cdot 0.00387 \cdot x_S \cdot x_B + 2 \cdot 0.00021 \cdot x_S \cdot x_M + 0.01112 \cdot x_B^2 - 2 \cdot 0.00020 \cdot x_B \cdot x_M + 0.00115 \cdot x_M^2}{1/2}
\end{align*}
\]
The Optimal Risky Portfolio

Solution

\[ \Sigma = \begin{bmatrix}
0.02778 & 0.00387 & 0.00021 \\
0.00387 & 0.01112 & -0.00020 \\
0.00021 & -0.00020 & 0.00115 \\
\end{bmatrix} \]

\[ \mu = \begin{bmatrix}
0.1073 \\
0.0737 \\
0.0627 \\
\end{bmatrix} \]

So the program of optimal risky portfolio is

\[
\max \quad \frac{0.1073 \cdot x_S + 0.0737 \cdot x_B + 0.0627 \cdot x_M - 0.03}{\left( 0.02778 \cdot x_S^2 + 2 \cdot 0.00387 \cdot x_S \cdot x_B + 2 \cdot 0.00021 \cdot x_S \cdot x_M + 0.01112 \cdot x_B^2 - 2 \cdot 0.00020 \cdot x_B \cdot x_M + 0.00115 \cdot x_M^2 \right)^{1/2}} \\
x_S + x_B + x_M = 1.
\]
The Optimal Risky Portfolio

\[
\mu - r_f \cdot e = \begin{pmatrix} 0.1073 \\ 0.0737 \\ 0.0627 \end{pmatrix} - \begin{pmatrix} 0.03 \\ 0.03 \\ 0.03 \end{pmatrix} = \begin{pmatrix} 0.0773 \\ 0.0437 \\ 0.0327 \end{pmatrix}
\]

We can find \( \kappa \) and \( y \) as below.

\[
\kappa = 1 \begin{pmatrix} 0.0773 \\ 0.0437 \\ 0.0327 \end{pmatrix} \cdot \begin{pmatrix} x_S \\ x_B \\ x_M \end{pmatrix} = 1 \cdot 0.0773 \cdot x_S + 0.0437 \cdot x_B + 0.0327 \cdot x_M
\]

\[
y = \begin{pmatrix} y_S \\ y_B \\ y_M \end{pmatrix} = \begin{pmatrix} x_S \\ 0.0773 \cdot x_S + 0.0437 \cdot x_B + 0.0327 \cdot x_M \end{pmatrix}
\]
The Optimal Risky Portfolio

Solution

\[ \mu - r_f \cdot e = \begin{bmatrix} 0.1073 \\ 0.0737 \\ 0.0627 \end{bmatrix} - \begin{bmatrix} 0.03 \\ 0.03 \\ 0.03 \end{bmatrix} = \begin{bmatrix} 0.0773 \\ 0.0437 \\ 0.0327 \end{bmatrix} \]

We can find \( \kappa \) and \( y \) as below.

\[
\kappa = 1 \left( \begin{bmatrix} 0.0773 \\ 0.0437 \\ 0.0327 \end{bmatrix} \right) \cdot \begin{bmatrix} x_S \\ x_B \\ x_M \end{bmatrix} = 1 \cdot 0.0773 \cdot x_S + 0.0437 \cdot x_B + 0.0327 \cdot x_M
\]

\[
y = \begin{bmatrix} y_S \\ y_B \\ y_M \end{bmatrix} = \begin{bmatrix} 0.0773 \cdot x_S + 0.0437 \cdot x_B + 0.0327 \cdot x_M \\ 0.0773 \cdot x_S + 0.0437 \cdot x_B + 0.0327 \cdot x_M \\ 0.0773 \cdot x_S + 0.0437 \cdot x_B + 0.0327 \cdot x_M \end{bmatrix}
\]
The Optimal Risky Portfolio

Solution

\[
\mu - r_f \cdot e = \begin{bmatrix} 0.1073 \\ 0.0737 \\ 0.0627 \end{bmatrix} - \begin{bmatrix} 0.03 \\ 0.03 \\ 0.03 \end{bmatrix}
\]
The Optimal Risky Portfolio

Solution

\[ \mu - r_f \cdot e = \begin{bmatrix} 0.1073 \\ 0.0737 \\ 0.0627 \end{bmatrix} - \begin{bmatrix} 0.03 \\ 0.03 \\ 0.03 \end{bmatrix} = \begin{bmatrix} 0.0773 \\ 0.0437 \\ 0.0327 \end{bmatrix} \]
The Optimal Risky Portfolio

Solution

\[
\mu - r_f \cdot e = \begin{bmatrix} 0.1073 \\ 0.0737 \\ 0.0627 \end{bmatrix} - \begin{bmatrix} 0.03 \\ 0.03 \\ 0.03 \end{bmatrix} = \begin{bmatrix} 0.0773 \\ 0.0437 \\ 0.0327 \end{bmatrix}
\]

We can find \( \kappa \) and \( y \) as below.
The Optimal Risky Portfolio

Solution

\[
\mu - r_f \cdot e = \begin{bmatrix} 0.1073 \\ 0.0737 \\ 0.0627 \end{bmatrix} - \begin{bmatrix} 0.03 \\ 0.03 \end{bmatrix} = \begin{bmatrix} 0.0773 \\ 0.0437 \\ 0.0327 \end{bmatrix}
\]

We can find \( \kappa \) and \( y \) as below.

\[
\kappa = \frac{1}{\left( \begin{bmatrix} 0.0773 \\ 0.0437 \\ 0.0327 \end{bmatrix} \right)^T \begin{bmatrix} x_S \\ x_B \\ x_M \end{bmatrix}}
\]
The Optimal Risky Portfolio

Solution

\[
\mu - r_f \cdot e = \begin{bmatrix} 0.1073 \\ 0.0737 \\ 0.0627 \end{bmatrix} - \begin{bmatrix} 0.03 \\ 0.03 \\ 0.03 \end{bmatrix} = \begin{bmatrix} 0.0773 \\ 0.0437 \\ 0.0327 \end{bmatrix}
\]

We can find \( \kappa \) and \( y \) as below.

\[
\kappa = \frac{1}{\begin{bmatrix} 0.0773 \\ 0.0437 \\ 0.0327 \end{bmatrix}^T \begin{bmatrix} x_S \\ x_B \\ x_M \end{bmatrix}} = \frac{1}{0.0773 \cdot x_S + 0.0437 \cdot x_B + 0.0327 \cdot x_M}
\]
The Optimal Risky Portfolio

Solution

\[
\mu - r_f \cdot e = \begin{bmatrix} 0.1073 \\ 0.0737 \\ 0.0627 \end{bmatrix} - \begin{bmatrix} 0.03 \\ 0.03 \\ 0.03 \end{bmatrix} = \begin{bmatrix} 0.0773 \\ 0.0437 \\ 0.0327 \end{bmatrix}
\]

We can find \( \kappa \) and \( y \) as below.

\[
\kappa = \frac{1}{\begin{bmatrix} 0.0773 \\ 0.0437 \\ 0.0327 \end{bmatrix}^T \begin{bmatrix} x_S \\ x_B \\ x_M \end{bmatrix}^*} = \frac{1}{0.0773 \cdot x_S + 0.0437 \cdot x_B + 0.0327 \cdot x_M}
\]

\[
y = \begin{bmatrix} y_S \\ y_B \\ y_M \end{bmatrix}
\]
### The Optimal Risky Portfolio

**Solution**

\[
\begin{align*}
\mu - r_f \cdot e & =
\begin{bmatrix}
0.1073 \\
0.0737 \\
0.0627
\end{bmatrix}
- 
\begin{bmatrix}
0.03 \\
0.03 \\
0.03
\end{bmatrix}
= 
\begin{bmatrix}
0.0773 \\
0.0437 \\
0.0327
\end{bmatrix}
\end{align*}
\]

We can find \( \kappa \) and \( y \) as below.

\[
\kappa = \frac{1}{\begin{bmatrix}
0.0773 \\
0.0437 \\
0.0327
\end{bmatrix}^T \cdot \begin{bmatrix}
x_S \\
x_B \\
x_M
\end{bmatrix}} = \frac{1}{0.0773 \cdot x_S + 0.0437 \cdot x_B + 0.0327 \cdot x_M}
\]

\[
y = \begin{bmatrix}
y_S \\
y_B \\
y_M
\end{bmatrix}
= \begin{bmatrix}
x_S \\
x_B \\
x_M
\end{bmatrix}
\begin{bmatrix}
x_S \\
x_B \\
x_M
\end{bmatrix}
= 
\begin{bmatrix}
\frac{x_S}{0.0773 \cdot x_S + 0.0437 \cdot x_B + 0.0327 \cdot x_M} \\
\frac{x_B}{0.0773 \cdot x_S + 0.0437 \cdot x_B + 0.0327 \cdot x_M} \\
\frac{x_M}{0.0773 \cdot x_S + 0.0437 \cdot x_B + 0.0327 \cdot x_M}
\end{bmatrix}
\]
Mean-Variance Optimization
Brief mention of other MVO models
**Maximizing the Sharpe Ratio**
More Topics not covered
References

The Optimal Risky Portfolio

And the equivalent quadratic programming problem is

$$\begin{align*}
\text{min } & \quad 0.02778 \cdot y^2_S + 2 \cdot 0.00387 \cdot y_S \cdot y_B + 2 \cdot 0.00021 \cdot y_S \cdot y_M + 0.01112 \cdot y^2_B - 2 \cdot 0.00020 \cdot y_B \cdot y_M + 0.00115 \cdot y^2_M - 0.0773 \cdot y_S + 0.0437 \cdot y_B + 0.0327 \cdot y_M - 1
\end{align*}$$

$$y_S + y_B + y_M - \kappa = 0$$

(Z. Donovan and M. Xu)
The Optimal Risky Portfolio

Solution

And the equivalent quadratic programming problem is

\[
\begin{align*}
    \text{min} & \quad 0.02778 \cdot y^2_S + 2 \cdot 0.00387 \cdot y_S \cdot y_B + 2 \cdot 0.00021 \cdot y_S \cdot y_M + 0.01112 \cdot y_B^2 \\
    & - 2 \cdot 0.00020 \cdot y_B \cdot y_M + 0.00115 \cdot y_M^2 \\
    & - y_S \kappa - y_B \kappa - y_M \kappa = 0.0773 \cdot y_S + 0.0437 \cdot y_B + 0.0327 \cdot y_M = 1 \quad (or y_S + y_B + y_M - \kappa = 0) \\
\end{align*}
\]
Solution
And the equivalent quadratic programming problem is
The Optimal Risky Portfolio

Solution

And the equivalent quadratic programming problem is

\[
\min \quad 0.02778 \cdot y_S^2 + 2 \cdot 0.00387 \cdot y_S \cdot y_B + 2 \cdot 0.00021 \cdot y_S \cdot y_M \\
+ 0.01112 \cdot y_B^2 - 2 \cdot 0.00020 \cdot y_B \cdot y_M + 0.00115 \cdot y_M^2
\]
The Optimal Risky Portfolio

Solution

And the equivalent quadratic programming problem is

\[
\begin{align*}
\text{min} & \quad 0.02778 \cdot y_S^2 + 2 \cdot 0.00387 \cdot y_S \cdot y_B + 2 \cdot 0.00021 \cdot y_S \cdot y_M \\
& \quad + 0.01112 \cdot y_B^2 - 2 \cdot 0.00020 \cdot y_B \cdot y_M + 0.00115 \cdot y_M^2 \\
\frac{y_S}{\kappa} + \frac{y_B}{\kappa} + \frac{y_M}{\kappa} & = 1 \quad (\text{or} \ y_S + y_B + y_M - \kappa = 0)
\end{align*}
\]
The Optimal Risky Portfolio

Solution

And the equivalent quadratic programming problem is

\[
\begin{align*}
\min & \quad 0.02778 \cdot y_S^2 + 2 \cdot 0.00387 \cdot y_S \cdot y_B + 2 \cdot 0.00021 \cdot y_S \cdot y_M \\
& \quad + 0.01112 \cdot y_B^2 - 2 \cdot 0.00020 \cdot y_B \cdot y_M + 0.00115 \cdot y_M^2 \\
& \quad \frac{y_S}{\kappa} + \frac{y_B}{\kappa} + \frac{y_M}{\kappa} = 1 \quad (\text{or } y_S + y_B + y_M - \kappa = 0) \\
& \quad 0.0773 \cdot y_S + 0.0437 \cdot y_B + 0.0327 \cdot y_M = 1
\end{align*}
\]
The Optimal Risky Portfolio

Solution

And the equivalent quadratic programming problem is

\[
\begin{align*}
\min & \quad 0.02778 \cdot y_S^2 + 2 \cdot 0.00387 \cdot y_S \cdot y_B + 2 \cdot 0.00021 \cdot y_S \cdot y_M \\
& \quad + 0.01112 \cdot y_B^2 - 2 \cdot 0.00020 \cdot y_B \cdot y_M + 0.00115 \cdot y_M^2 \\
\frac{y_S}{\kappa} + \frac{y_B}{\kappa} + \frac{y_M}{\kappa} &= 1 \quad \text{(or } y_S + y_B + y_M - \kappa = 0) \\
0.0773 \cdot y_S + 0.0437 \cdot y_B + 0.0327 \cdot y_M &= 1 \\
y, \kappa &\geq 0
\end{align*}
\]
Topics not covered

- Returns-Based Style Analysis
Topics not covered

- Returns-Based Style Analysis
- Recovering Risk-Neutral Probabilities from Options Prices
References


References


Z. Donovan and M. Xu, Optimization Methods in Finance
References

References


2. Website: http://www.investopedia.com/terms/s/sharperatio.asp
References


2. Website: http://www.investopedia.com/terms/s/sharperatio.asp

3. Website: http://mirceatrandafir.com/teaching/econ435/
## References


