Robust Optimization: Applications

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April 30, 2015
1  Overview
   • Theory of Robust Optimization
   • Applications of Robust Optimization
Outline

1. Overview
   - Theory of Robust Optimization
   - Applications of Robust Optimization

2. Applications
   - Shortest Path Problem
   - Facility Location Problem
   - Portfolio Selection Problem
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Chinese proverb

To be uncertain is to be uncomfortable, but to be certain is to be ridiculous.
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Reasoning
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- The data of real-world optimization problems often are not known exactly at the time the problem is being solved.
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- In real-world applications of optimization, even a small uncertainty in the data can make the nominal optimal solution to the problem completely meaningless from a practical viewpoint.
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Reasoning
- The data of real-world optimization problems often are not known exactly at the time the problem is being solved.
- In real-world applications of optimization, even a small uncertainty in the data can make the nominal optimal solution to the problem completely meaningless from a practical viewpoint.
- Consequently, in optimization, there exist a real need of a methodology capable of detecting cases when data uncertainty can heavily effect the quality of the nominal solution.
<table>
<thead>
<tr>
<th>Note</th>
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</table>

Optimization based on nominal values often lead to SEVERE infeasibilities.
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Example

Consider the constraint:

\[ 3.000 \cdot x_1 - 2.999 \cdot x_2 \leq 1 \]

Suppose:
\[ x_1 = x_2 = 1000 \] is optimal, and the number 3.000 is certain.

Then LHS of constraint can be:

\[ 3.000 \cdot 1000 - 2.999 \cdot 1000 \leq 1 \]

Now suppose:
\[ x_1 = x_2 = 1000 \] is optimal, the number 3.000 is uncertain and maximal deviation is only 1 percent.

Then LHS of constraint can be:

\[ 3.030 \cdot 1000 - 2.999 \cdot 1000 = 31 > 1 \]
Example

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Stochastic Programming Vs. Robust Optimization

Stochastic Programming:
- Often difficult to specify reliably the distribution of uncertain data.
- Expectations/probabilities often difficult to calculate.

Robust Optimization:
- Does not assume stochastic nature of the uncertain data.
- Remains computationally tractable.
- Sometimes is too pessimistic.
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Overview

Applications

Theory of Robust Optimization

Applications of Robust Optimization

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LP Programming with uncertainty
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- Standard LP form:

\[
\begin{align*}
\min z &= c \cdot x \\
A \cdot x &\leq b
\end{align*}
\]
LP Programming with uncertainty

- Standard LP form:

\[ \min z = c \cdot x \]
\[ A \cdot x \leq b \]

- Uncertain LP form:

\[ \min z = \tilde{c} \cdot x \]
\[ \tilde{A} \cdot x \leq \tilde{b} \]
Uncertain Linear Constrain
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- Certain Linear Constrain:

\[ A \cdot x \leq b \]
**Uncertain Linear Constraint**

- **Certain Linear Constraint:**
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- Uncertain Linear Constrain:
  \[ \mathbf{	ilde{A}} \cdot \mathbf{x} \leq \mathbf{b} \]

- Chance Constrain:
  \[ P(\mathbf{	ilde{A}} \cdot \mathbf{x} > \mathbf{b}) \leq \mathcal{E} \]
Affine Uncertainty

$\tilde{A} = A(\tilde{z}) = A_0 + N \sum_{j=1}^{\text{N factors}} A_j \cdot \tilde{z}_j$

where $N$ is the number of factors which comprise the uncertainty, and $|z| \leq 1$.

Now, replacing the uncertain parameter with the above equation:

$(A_0 + N \sum_{j=1}^{\text{N factors}} A_j \cdot \tilde{z}_j) \cdot x \leq b$
Affine Uncertainty

\[ \tilde{A} = A(\tilde{z}) = A^0 + \sum_{j=1}^{N} A_j \cdot \tilde{z}_j \]
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\[ (A^0 + \sum_{j=1}^{N} A_j \cdot \tilde{z}_j) \cdot x \leq b \]
<table>
<thead>
<tr>
<th><strong>Budget of Uncertainty ($\Omega$)</strong></th>
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Budget of Uncertainty ($\Omega$)

$$
A(\tilde{z}_j) \cdot x \leq b \\
\forall \tilde{z}_j \in U\Omega
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Box

\[ \sum_{j=1}^{N} \tilde{z}_j \leq \Omega \]
Budget of Uncertainty ($\Omega$)

\[ A(\tilde{z}_j) \cdot x \leq b \]
\[ \forall \tilde{z}_j \in U_\Omega \]

- **Box LP**
  \[ \sum_{j=1}^{N} \tilde{z}_j \leq \Omega \]

- **Ellipsodial CQP**
  \[ \sum_{j=1}^{N} \tilde{z}_j^2 \leq \Omega \]
Budget of Uncertainty (Ω)

Measure of the level of robustness. Used to adjust conservativeness of the robust optimization solution. $\Omega \in (0, \sqrt{N})$.

$\Omega = 0$ means no uncertainty and $\Omega = \sqrt{N}$ is worst case budget.

$E = \exp(-\Omega^2/2)$
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$$\mathcal{E} = \exp(-\frac{\Omega^2}{2})$$
Example

$$E = \exp(-0.2^2) = 1.000$$

$$E = \exp(-1.2^2) = 0.606$$

$$E = \exp(-2.2^2) = 0.135$$

$$E = \exp(-3.2^2) = 0.011$$
Example

\[ E = \exp\left(-\frac{0^2}{2}\right) = 1.000 \]
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Applications of Robust Optimization

- **Mining engineering:**
  - Production scheduling: uncertainty of grade and price;
  - Mineral processing circuit design: uncertainty of grade, feed rate and price.

- **Civil Engineering:**
  - Shortest path problem;
  - Facility locations: uncertainty of demand.

- **Electrical Engineering:**
  - Circuit design: minimizing delay in digital circuits when the underlying gate delays are not known exactly.

- **Finance:**
  - Robust portfolio optimization: random returns;
  - Robust risk management: uncertainty of Value at Risk.
Applications

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Example

Given a weighted digraph, find the shortest path from 1 to 3 if:

- Link travel costs \(c_j\) are subject to bounded uncertainty;
- There are two sources of uncertainty \(\tilde{z}_j\).
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\[ (c_j) \]

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![Diagram showing weighted digraph with costs $C_1 = 2$, $C_2 = 4$, $C_3 = 3$, and $C_4 = 2$.]
Example

\[ \tilde{c}_1 = 2 + 1.5 \tilde{z}_1 + 1.5 \tilde{z}_2 \]
\[ \tilde{c}_2 = 4 + 0 \tilde{z}_1 + 0 \tilde{z}_2 \]
\[ \tilde{c}_3 = 3 + 0 \tilde{z}_1 + 0 \tilde{z}_2 \]
\[ \tilde{c}_4 = 2 + 1 \tilde{z}_1 + 1 \tilde{z}_2 \]

where:

\[ \tilde{z}_1 \leq \Omega \leq \Omega \in (0, 1, 2) \]
Example

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Example

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\[ \tilde{c}_2 = 4 + 0.5\tilde{z}_1 + 0.5\tilde{z}_2 \]
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\[ \tilde{c}_2 = 4 + 0.5\tilde{z}_1 + 0.5\tilde{z}_2 \]

\[ \tilde{c}_3 = 3 + 0.5\tilde{z}_1 + 0.5\tilde{z}_2 \]
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>[ \tilde{c}_1 = 2 + 1.5\tilde{z}_1 + 1.5\tilde{z}_2 ]</td>
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where:

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\[ \Omega \in (0, 1, 2) \]
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\end{align*}
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where:

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\tilde{z}_1^2 + \tilde{z}_2^2 \leq \Omega \\
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<table>
<thead>
<tr>
<th>(\Omega=0)</th>
<th>(\tilde{y}_{k1})</th>
<th>(\tilde{y}_{k1})</th>
<th>ROBUST SP</th>
<th>CLASSICAL SP</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-2</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>1-4</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>3-2</td>
<td>0</td>
<td>0</td>
<td>7</td>
<td>7</td>
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<tr>
<td>1-2</td>
<td>$\sqrt{2}/2$</td>
<td>$\sqrt{2}/2$</td>
<td>8.83</td>
<td>6</td>
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<tr>
<td>1-4</td>
<td>$\sqrt{2}/2$</td>
<td>$\sqrt{2}/2$</td>
<td>8.24</td>
<td>4</td>
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<tr>
<td>3-2</td>
<td>$\sqrt{2}/2$</td>
<td>$\sqrt{2}/2$</td>
<td>9.82</td>
<td>7</td>
</tr>
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<td>1</td>
<td>10.00</td>
<td>6</td>
</tr>
<tr>
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<td>1</td>
<td>1</td>
<td>10.00</td>
<td>4</td>
</tr>
<tr>
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Application of Facility Location Models

- New airport, hospital, and school.
Application of Facility Location Models

- New airport, hospital, and school.
- Addition of a new workstation.
Application of Facility Location Models

- New airport, hospital, and school.
- Addition of a new workstation.
- Warehouse location.
Application of Facility Location Models

- New airport, hospital, and school.
- Addition of a new workstation.
- Warehouse location.
- Bathroom location in a facility etc.
The goal of the facility location problem in the airline industry is to find an optimal location of hubs where demand is uncertain and its distribution is not fully specified.
Overview

Applications

Shortest Path Problem
Facility Location Problem
Portfolio Selection Problem

Example

- The goal of the facility location problem for airline industry is to find an optimal location of hubs.
Example

- The goal of the facility location problem for airline industry is to find an optimal location of hubs.
- Where demand is uncertain and its distribution is not fully specified.
Example

(a): Base hub location
Total Cost: 1336.19
Example

(b): Robust hub location, $\Omega = 1.5$
Total Cost: 1450.58
Example

(c): Robust hub location, $\Omega = 2$
Total Cost: 1450.58
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### Example

We consider the following simple portfolio optimization example:

Maximize $\mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3$

subject to

$TE(x_1, x_2, x_3) \leq 0$.

$10 x_1 + x_2 + x_3 = 1$

$x_1, x_2, x_3 \geq 0$

where $x_1$ and $x_2$ are first and second assets. $x_3$ represents proportion of the funds that are not invested. The benchmark is the portfolio that invests funds half-and-half in the two assets. $TE(x)$ represents the tracking error of the portfolio with respect to the half-and-half benchmark.
Example

We consider the following simple portfolio optimization example:

Maximize $\mu_1 \cdot x_1 + \mu_2 \cdot x_2 + \mu_3 \cdot x_3$

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$$x_1, x_2, x_3 \geq 0$$

where

$x_1$ and $x_2$ are first and second assets.

$x_3$ represents proportion of the funds that are not invested.

The benchmark is the portfolio that invests funds half-and-half in the two assets.
Example

We consider the following simple portfolio optimization example:

Maximize \( \mu_1 \cdot x_1 + \mu_2 \cdot x_2 + \mu_3 \cdot x_3 \)

subject to

\[
TE(x_1, x_2, x_3) \leq 0.10
\]
\[
x_1 + x_2 + x_3 = 1
\]
\[
x_1, x_2, x_3 \geq 0
\]

where

\( x_1 \) and \( x_2 \) are first and second assets.
\( x_3 \) represents proportion of the funds that are not invested.

The benchmark is the portfolio that invests funds half-and-half in the two assets. \( TE(x) \) represents the tracking error of the portfolio with respect to the half-and-half benchmark.
Example

Tracking errors are reported as a ”standard deviation percentage” difference. This measure reports the difference between the return an investor receives and that of the benchmark he or she was attempting to imitate.
Example

Tracking errors are reported as a "standard deviation percentage" difference. This measure reports the difference between the return an investor receives and that of the benchmark he or she was attempting to imitate.

$$TE = \sqrt{\begin{bmatrix} x_1 - 0.5 \\ x_2 - 0.05 \\ x_3 \end{bmatrix}^T \begin{bmatrix} 0.1764 & 0.09702 & 0 \\ 0.09702 & 0.1089 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 - 0.5 \\ x_2 - 0.05 \\ x_3 \end{bmatrix}}$$
Example

**Figure:** The feasible set of the portfolio selection problem
We now build a relative robustness model for this portfolio problem:

**Scenario 1:** $(\mu_1, \mu_2, \mu_3) : (6, 4, 0)$

**Scenario 2:** $(\mu_1, \mu_2, \mu_3) : (5, 5, 0)$

**Scenario 3:** $(\mu_1, \mu_2, \mu_3) : (4, 6, 0)$

So, the objective values will be:

**Scenario 1:** 5.662

**Scenario 2:** 5.662

**Scenario 3:** 5.000
Example

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Example

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Example

Relative robust formulation:
Example

Relative robust formulation:

$$\min_{x,t} t$$
Example

Relative robust formulation:

\[
\min_{x,t} \quad t \\
5.662 - (6x_1 + 4x_2) \leq t
\]
Example

Relative robust formulation:

\[
\begin{align*}
\min_{x,t} & \quad t \\
5.662 - (6x_1 + 4x_2) & \leq t \\
5.662 - (4x_1 + 6x_2) & \leq t
\end{align*}
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Relative robust formulation:

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\min_{x,t} \quad t \\
5.662 - (6x_1 + 4x_2) \leq t \\
5.662 - (4x_1 + 6x_2) \leq t \\
5.000 - (5x_1 + 5x_2) \leq t \\
TE(x_1, x_2, x_3) \leq 0.10
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Example

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5.662 - (6x_1 + 4x_2) & \leq t \\
5.662 - (4x_1 + 6x_2) & \leq t \\
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x_1 + x_2 + x_3 & = 1
\end{align*}
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Example

Relative robust formulation:

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\min_{x,t} & \quad t \\
5.662 - (6x_1 + 4x_2) & \leq t \\
5.662 - (4x_1 + 6x_2) & \leq t \\
5.000 - (5x_1 + 5x_2) & \leq t \\
TE(x_1, x_2, x_3) & \leq 0.10 \\
x_1 + x_2 + x_3 & = 1 \\
x_1, x_2, x_3 & \geq 0
\end{align*}
\]
Example

choosing a maximum tolerable regret level of 0.75 we get the following feasibility problem:

\[
\begin{align*}
5.662 - (6x_1 + 4x_2) &\leq 0.75 \\
5.000 - (5x_1 + 5x_2) &\leq 0.75 \\
TE(x_1, x_2, x_3) &\leq 0.10 \\
x_1 + x_2 + x_3 &= 1 \\
x_1, x_2, x_3 &\geq 0
\end{align*}
\]
Example

choosing a maximum tolerable regret level of 0.75 we get the following feasibility problem:

\[ \text{Find } x \]

\[ 5.662 - (6x_1 + 4x_2) \leq 0.75 \]

\[ 5.000 - (4x_1 + 5x_2) \leq 0.75 \]

\[ \text{TE}(x_1, x_2, x_3) \leq 0.10 \]

\[ x_1 + x_2 + x_3 = 1 \]

\[ x_i \geq 0 \]
Example

choosing a maximum tolerable regret level of 0.75 we get the following feasibility problem:

Find \( x \)

\[
5.662 - (6x_1 + 4x_2) \leq 0.75
\]
Example

choosing a maximum tolerable regret level of 0.75 we get the following feasibility problem:

Find $x$

\[
\begin{align*}
5.662 - (6x_1 + 4x_2) &\leq 0.75 \\
5.662 - (4x_1 + 6x_2) &\leq 0.75
\end{align*}
\]
Example

choosing a maximum tolerable regret level of 0.75 we get the following feasibility problem:

\[
\text{Find } x \\
5.662 - (6x_1 + 4x_2) \leq 0.75 \\
5.662 - (4x_1 + 6x_2) \leq 0.75 \\
5.000 - (5x_1 + 5x_2) \leq 0.75
\]
Example

choosing a maximum tolerable regret level of 0.75 we get the following feasibility problem:

\[
\begin{align*}
\text{Find} & \quad x \\
5.662 - (6x_1 + 4x_2) & \leq 0.75 \\
5.662 - (4x_1 + 6x_2) & \leq 0.75 \\
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TE(x_1, x_2, x_3) & \leq 0.10
\end{align*}
\]
Example

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\[
5.662 - (6x_1 + 4x_2) \leq 0.75 \\
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TE(x_1, x_2, x_3) \leq 0.10 \\
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Example

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\text{TE}(x_1, x_2, x_3) & \leq 0.10 \\
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x_1, x_2, x_3 & \geq 0
\end{align*}
\]
Example

Figure: Set of solutions with regret less than 0.75