Robust Optimization: Theory and Tools

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Outline

1. Theory of Robust Optimization
   - Introduction
   - Uncertainty sets
   - RO Concepts
   - Summary
   - Example

2. Adjustable Robust Optimization (ARO)
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2 Adjustable Robust Optimization (ARO)
   - Tools and Strategies for Robust Optimization
   - Sampling
   - Saddle-Point Characterizations
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3. Bibliography
Robust Optimization refers to the modeling of optimization problems with data uncertainty. Robust optimization models can be useful in the following situations:

- Some of the problem parameters are estimates and carry estimation risk.
- There are constraints with uncertain parameters that must be satisfied regardless of the values of these parameters.
- The objective function or the optimal solutions are sensitive to perturbations.
- The decision-maker cannot afford to low-probability but high-magnitude risks.

M. Mera Trujillo
Optimization Methods in Finance
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Definition (Robust Optimization (RO))

Robust Optimization (RO) is a complementary methodology to stochastic programming and sensitivity analysis. It seeks a solution that will have an "acceptable" performance under most realizations of the uncertain inputs. Usually, no distribution assumption is made on uncertain parameters (if such information is available, it can be utilized beneficially). Usually, it is a conservative (worst-case oriented) methodology.

Robust Optimization is useful if:

- Some parameters come from an estimation process and may be contaminated with estimation errors.
- There are "hard" constraints that must be satisfied no matter what.
- The objective function value/optimal solutions are highly sensitive to perturbations.
- The modeler/designer cannot afford low probability high-magnitude risks (typical example: designing a bridge).
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- Kouvelis and Yu (minimax regret, Robust Discrete Optimization) [1].
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**Major problems:**

- How to represent uncertainty?
- How to compute a robust solution?
- What is a robust solution anyway?

We want to find a **optimal (feasible) solution**. An optimal solution is robust if it minimizes maximum relative regret.
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  We want to find a **optimal (feasible) solution**.
  
  An optimal solution is robust if it minimizes maximum relative regret.
Maximum relative regret

What is maximum relative regret?

Example: Consider the shortest path problem on a directed graph where arc costs are subject to uncertainty. Let us assume arc \((i,j)\) can have as cost value any value in the interval \([\bar{c}_{ij}, \tilde{c}_{ij}]\).

Define the maximum relative regret associated with any path: Put all arc costs on the path to their upper bounds and all other arc costs to their lower bounds. Find the shortest path in this realization of arc costs. Maximum Relative Regret \(=\) the cost of the path (at lower bounds), that of the shortest path.
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Definition (Uncertainty)

Data uncertainty or uncertainty in the parameters is described through uncertainty sets that contain many possible values that may be realized for the uncertain parameters. The size of the uncertainty set is determined by the level of desired robustness.
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Uncertainty sets can represent or may be formed by differences of opinions on future values of certain parameters, alternative estimates of parameters generated via statistical techniques from historical data and/or Bayesian techniques, among other things.

Types of uncertainty sets

Common types of uncertainty sets encountered in robust optimization models include the following:

- Uncertainty sets representing a finite number of scenarios generated for the possible values of the parameters:
  \[ U = \{ p_1, p_2, \ldots, p_k \} \]
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Types of uncertainty sets

- Uncertainty sets representing an interval description for each uncertain parameter:

\[ U = \{ p : l \leq p \leq u \} \]

Confidence intervals encountered frequently in statistics can be the source of such uncertainty sets.

Ellipsoidal uncertainty sets:

\[ U = \{ p : p = p_0 + M \cdot u, ||u|| \leq 1 \} \]

These uncertainty sets can also arise from statistical estimation in the form of confidence regions.
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To determine the uncertainty set that is appropriate for a particular model as well as the type of uncertainty sets that lead to tractable problems, it is a non-trivial task. As a general guideline, the shape of the uncertainty set will often depend on the sources of uncertainty as well as the sensitivity of the solutions to these uncertainties.
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- As a general guideline, the shape of the uncertainty set will often depend on the sources of uncertainty as well as the sensitivity of the solutions to these uncertainties.
The size of the uncertainty set, on the other hand, will often be chosen based on the desired level of robustness. For example, in mean-variance portfolio optimization, uncertain parameters reflect the “true” values of moments of random variables, there is no way of knowing these unobservable true values exactly. In such cases, after making some assumptions about the stationarity of these random processes, we can generate estimates of these true parameters using statistical procedures.
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Example:

Using a linear factor model for the multivariate returns of several assets and estimate the factor loading matrices via linear regression, the confidence regions generated for these parameters are ellipsoidal sets and they advocate their use in robust portfolio selection as uncertainty sets (Goldfarb and Iyengar).

To generate interval type uncertainty sets, use bootstrapping strategies as well as moving averages of returns from historical data (Tütüncü and Koenig).

NOTE: The shape and the size of the uncertainty set can significantly affect the robust solutions generated. However, with few guidelines backed by theoretical and empirical studies, their choice remains an art form at the moment.
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Using a linear factor model for the multivariate returns of several assets and estimate the factor loading matrices via linear regression, the confidence regions generated for these parameters are ellipsoidal sets and they advocate their use in robust portfolio selection as uncertainty sets (Goldfarb and Iyengar).

To generate interval type uncertainty sets, use bootstrapping strategies as well as moving averages of returns from historical data (Tütüncü and Koenig).

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The shape and the size of the uncertainty set can significantly affect the robust solutions generated.
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The shape and the size of the uncertainty set can significantly affect the robust solutions generated. However with few guidelines backed by theoretical and empirical studies, their choice remains an art form at the moment.
Outline

1. Theory of Robust Optimization
   - Introduction
   - Uncertainty sets
   - RO Concepts
   - Summary
   - Example

2. Tools and Strategies for Robust Optimization
   - Adjustable Robust Optimization (ARO)
   - Sampling
   - Saddle-Point Characterizations

3. Bibliography
Constraint Robustness

This refers to situations where the uncertainty is in the constraints and we seek solutions that remain feasible for all possible values of the uncertain inputs. For example: multi-stage problems where the uncertain outcomes of earlier stages have an effect on the decisions of the later stages and the decision variables must be chosen to satisfy certain balance constraints. (e.g., inputs to a particular stage cannot exceed the outputs of the previous stage) no matter what happens with the uncertain parameters of the problem. The solution must be constraint-robust with respect to the uncertainties of the problem.
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Mathematical model for finding constraint-robust solutions:

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\min_{x} \ f(x)
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\[
G(x, p) \in K
\]

where \(x\) are the decision variables, \(f\) is the (certain) objective function, \(G\) and \(K\) are the structural elements of the constraints that are assumed to be certain and \(p\) are the possibly uncertain parameters of the problem.
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$$G(x, p) \in K, \forall p \in \mathcal{U}$$
Robust feasible set

The robust feasible set is the intersection of the feasible sets $S(p) = \{x : G(x, p) \in K\}$ indexed by the uncertainty set $\mathcal{U}$.
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For an ellipsoidal feasible set with $\mathcal{U} = \{p_1, p_2, p_3, p_4\}$, where $p_i$ correspond to the uncertain center of the ellipse.
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Figure: Constraint robustness
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The feasible set $S$ is the real line, the uncertainty set is $\mathcal{U} = \{p_1, p_2, p_3, p_4, p_5\}$, and the objective function $f(x, p_i)$ is a convex quadratic function whose parameters $p_i$ determine its shape.
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What are we dealing with?

Consider the optimization problem:

$$\min \ f(x, \xi)$$

subject to

$$g_i(x, \xi) \leq X$$

$x$ is the vector of variables, $\xi$ is the vector of data (uncertain), $f$ and $g_i$ are convex functions, and $X$ is a possibly non-convex (e.g., the set of nonnegative integers, or binary integers) set.
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Our Typical Optimization Problems

Linear Programming:

\[
\min_{x} \mathbf{c}^T \cdot x \\
\text{subject to } A \cdot x \geq b
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\(A, b, \text{ and } c\) could be plagued with uncertainty, or could be just estimates from a simulation or discretization process.
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Quadratic Programming:

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\min \left( x^2 \right) \cdot x + c^T \cdot x
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Ben-Tal and Nemirovski Approach to Robust Optimization

Consider the linear program:

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\begin{align*}
\min & \quad c^T \cdot x \\
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\end{align*}
\]

Assume each row \( a_i \) of \( A \) is uncertain but known to lie in ellipsoids.

\[ E_i = \{ a_i : a_i = \bar{a}_i + P_i \cdot u_i, ||u_i||^2 \leq 1 \} \]

where \( P_i \) is symmetric positive (semi)definite matrix for all \( i \).

Assume rows \( a_i \) assume values independently of one another.
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- Assume rows \(a_i\) assume values independently of one another.
What is a robust solution in the world of Ben-Tal and Nemirovski?

We want to make sure the constraints
\[ Ax \geq b \]
are satisfied for all realizations of the data. We do not tolerate a violation of the constraints for any values of the uncertain parameters in the uncertainty set. Among such solutions, pick one that minimizes the objective function value.
What is a robust solution in the world of Ben-Tal and Nemirovski?

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- We want to make sure the constraints $A \cdot x \geq b$ are satisfied for all realizations of the data.
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Consider a multi-period optimization problem with uncertain parameters where uncertainty is revealed progressively through periods.
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- A subset of the decision variables can be chosen after observing realizations of some uncertain parameters.
- This allows to correct the earlier decision made under a smaller information set.
- In stochastic programming, this is called “recourse”
Consider a multi-period optimization problem with uncertain parameters where uncertainty is revealed progressively through periods.

A subset of the decision variables can be chosen after observing realizations of some uncertain parameters.

This allows to correct the earlier decision made under a smaller information set.

In stochastic programming, this is called “recourse”

In robust optimization it is called “Adjustable Robust Optimization” (ARO).
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Due to Ben-Tal, Guslitzer and Nemirovski.
Adjustable robust optimization (ARO) formulations model these decision environment and allow recourse action. ARO models were recently introduced for uncertain linear programming problems. For example:

Consider the two-stage linear optimization problem given below whose first-stage decision variables $x_1$ need to be determined now, while the second-stage decision variables $x_2$ can be chosen after the uncertain parameters $A_1$, $A_2$, and $b$ are realized:

\[
\min x_1, x_2 \left\{ c^T x_1 : A_1 x_1 + A_2 x_2 \leq b \right\}.
\]
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Assume that $A^1$ is revealed to the modeler after choosing $x^1$. So, at the moment of choosing $x^2$ the modeler knows the value of $A^1$. 
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Could we make robust choices of $x^2$ taking this sequential nature of the decision process into account?
Adjustable Robustness (standard robust counterpart)

- Let $\mathcal{U}$ denote the uncertainty set for parameters $A^1$, $A^2$, and $b$. 

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The standard constraint robust optimization formulation for this problem seeks to find vectors $x_1$ and $x_2$ that optimize the objective function and satisfy the constraints of the problem for all possible realizations of the constraint coefficients. Both sets of variables must be chosen before the uncertain parameters can be observed and therefore cannot depend on these parameters.

The robust counterpart is (RC):

$$\min x_1 \{ c^T x_1 : \exists x_2 \forall (A_1, A_2, b) \in \mathcal{U}: A_1 \cdot x_1 + A_2 \cdot x_2 \leq b \}$$

Here notice that the choice of $x_2$ is independent of the realized values of the uncertain parameters.

This ignores the multi-stage nature of the decision process.
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- Because we do not know the functional form of the dependency.
- Only very simple uncertainty sets allow “nice” ARCs. Otherwise, we have to assume a simple functional form for dependencies, e.g., affine dependency.
An Example from Network Design

- Ordoñez and Zhao considered the following network capacity expansion problem subject to a budget constraint:

\[
\begin{align*}
\min & \quad c^T x \\
\text{subject to} & \quad N \cdot x = b, \\
& \quad x \leq u + y, \\
& \quad d^T \leq I,
\end{align*}
\]

Where the vector \( y \in \mathbb{R}^m \) denotes capacity expansion decisions, \( x \in \mathbb{R}^m \) the flow variables. The constraints \( N \cdot x = b \) represent the flow balance equations, \( c \) represents the transportation cost coefficients, and \( d \) the cost of incremental unit capacity. Assume \( c \) and \( b \) are uncertain, i.e., \( c \in U_c \) and \( b \in U_b \), where \( U_c \) and \( U_b \) are suitable (closed, bounded, convex) uncertainty sets.
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Adjustable Robust Counterpart

\[ \text{The Adjustable Robust Counterpart is:} \]
\[ z_{ARC} = \min_{y, \gamma} \]
\[ \text{subject to} \]
\[ d^T x \leq I, \quad y \geq 0, \quad \forall c \in U_c, \quad b \in U_b \]
\[ \exists x : \begin{cases} N \cdot x = b_0 \\ 0 \leq x \leq u + y \end{cases} \]
\[ C^T x \leq \gamma \]

Ordoñez and Zhao proved:
\[ z_{ARC} = \min_{y \geq 0} \]
\[ d^T y \leq I_{\max} \]
\[ c \in U_c, \quad b \in U_b \]
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Outline

1. Theory of Robust Optimization
   - Introduction
   - Uncertainty sets
   - RO Concepts
   - Summary
   - Example

2. Tools and Strategies for Robust Optimization
   - Adjustable Robust Optimization (ARO)
   - Sampling
   - Saddle-Point Characterizations

3. Bibliography
One of the simplest strategies for achieving robustness under uncertainty is to sample several scenarios for the uncertain parameters from a set that contains possible values of these parameters. Sampling can be done with or without using distributional assumptions on the parameters and produces a robust optimization formulation with a finite uncertainty set. If uncertain parameters appear in the constraints, we create a copy of each such constraint corresponding to each scenario. Uncertainty in the objective function can be handled in a similar manner.
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Uncertainty in the objective function can be handled in a similar manner.
Consider the generic uncertain optimization problem:

$$\min_x f(x) \quad G(x, p) \in K$$

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Optimization Methods in Finance
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If the uncertainty set $U$ is a finite set, i.e., $U = \{p_1, p_2, \ldots, p_k\}$, the robust formulation is obtained as follows:

$$\min_{t, x} \quad t - f(x, p_i) \geq 0, \quad i = 1, \ldots, k,$$

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The robust solution can be characterized using saddle-point conditions when the original problem satisfies certain convexity assumptions. The benefit of this characterization is that we can then use algorithms such as interior-point methods already developed and available for saddle-point problems.

Consider:

$$\min_{x \in S} \max_{p \in U} f(x, p)$$

Note that the dual of this robust optimization problem is obtained by changing the order of the minimization and maximization problems:

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Saddle-Point Characterizations

From standard results in convex analysis we have the following conclusion:

**Lemma:**

If $f(x, p)$ is a convex function of $x$ and concave function of $p$, if $S$ and $U$ are nonempty and at least one of them is bounded the optimal values of the problems above and there exists a saddle point $(x^*, p^*)$ such that:

$$f(x^*, p^*) \leq f(x^*, p_*) \leq f(x, p^*), \forall x \in S, p \in U.$$
Saddle-Point Characterizations

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\[
f(x^*, p^*) \leq f(x, p^*) \leq f(x, p) \leq f(x^*, p), \quad \forall x \in S, \quad p \in \mathcal{U}
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Saddle-Point Characterizations

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Software

Robust Optimization and Robust Programming software:

http://www.aimms.com/operations-research/mathematical-programming/robust-optimization/

Conclusions

Robustness = best solution against the worst possible data realization.
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