A Genetic Algorithm for Designing Constellations with Low Error Floors

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Outline

Introduction

- BICM is a standard approach for coding in modern wireless systems
- Performance of BICM system could be improved by using iterative decoding and demodulation
- Plots of BER for BICM-ID system exhibit waterfall region and error-floor region
- Performance of BER is determined by labeling map and choice of constellation
- Actual error floor can be approximated using EFF bound

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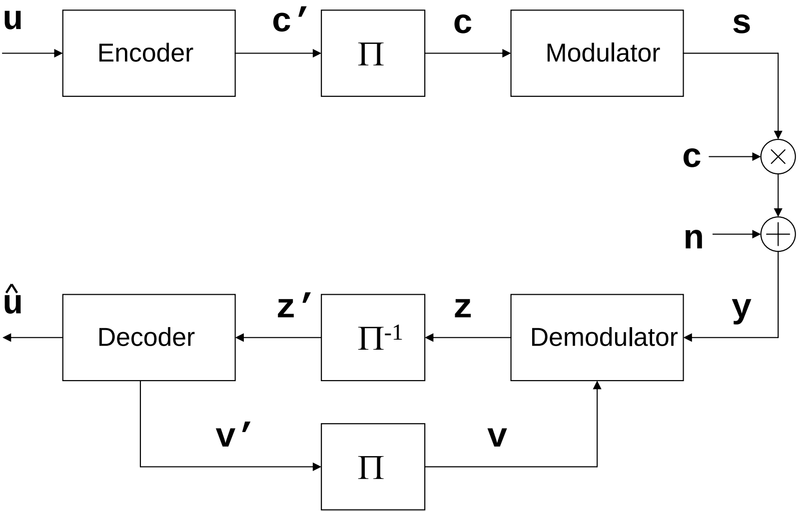
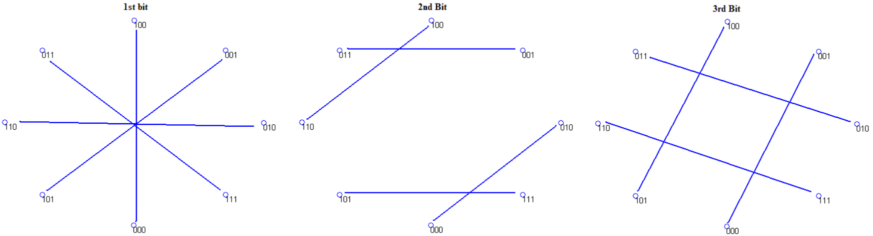
BICM-ID System Model



EFF(Error Free Feedback)Pairs



- x and $g_k(x)$ form an EFF pair

EFF BOUND

- Bit error probability for BICM-ID system is defined as follows

$$\log_{10} P_b \approx \frac{-d_f}{10}\left[\left(Rd_h^2\right)_{dB} + \left(\frac{\mathcal{E}_b}{N_0}\right)_{dB}\right] + \kappa_{dB}. \quad (1)$$

- Harmonic mean d_h^2

$$d_h^2 = \left[\frac{1}{m2^{m-1}}\sum_{k=0}^{m-1}\sum_{x' \in \mathcal{X}_k^{(1)}} ||x' - g_k(x')||^{-2}\right]^{-1} \quad (2)$$

- Choice of signal set $\mathcal{X}$ and symbol labelling map μ influence harmonic mean
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Label Mapping Optimization

- Harmonic mean could be maximized by minimizing following cost function which can be formulated as an instance of QAP(Quadratic Assignment Problem)[Huang and Ritcey:2005]

$$\min_{\boldsymbol{\mu}} \sum_{k=0}^{m-1} \sum_{x \in \mathcal{X}_k^{(1)}} ||x - g_k(x)||^{-2} \Leftrightarrow \min_{\boldsymbol{\mu}} \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} f_{i,j} d_{\mu(i),\mu(j)} \tag{3}$$

- Flow and distance matrix are defined as follows

$$f_{i,j} = \begin{cases} 1, & \mathcal{X}(i), \mathcal{X}(j) \text{ form an EFF pair} \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

$$d_{i,j} = \begin{cases} ||x_i - x_j||^{-2}, & i \neq j \\ 0, & i = j \end{cases} \tag{5}$$

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Different Techniques

- QAP is an NP Hard Problem
  - $16! = 2.09227899 * 10^{13}$
- Different techniques evolved to solve it
  - Reactive Tabu Search [R.Battiti,G.Tecchiolli:1994]
  - Branch and bound technique [Br1998, Mar1999]
  - Simulated Annealing [Nis:1995]
  - Genetic Algorithm [Y.Wu,P.Ji:2007]
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Pseudo Code for GA

```
Begin
    Choose N random mappings
    Sort the mappings from Best to Worst
    Pick first parent from anything other than Best
    Pick second parent that is better than first

    Apply breeding to selected parents
    Start mutation process if children worse than parent
    If not culling generation
        Replace worst parent with child if better
    else
        Replace worst among the population with best child
  Repeat the above process for certain number of generations
End
```

Breeding Process

- Two Parents are chosen

7	1	6	3	2	5	4	0		6	7	1	2	5	3	0	4

- Exchange cross over points from parents

X	7	X	X	X	3	X	X		X	1	X	X	X	5	X	X

- Copy down the elements from direct parents

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- Place the remaining elements in the unfilled positions

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Selection Process

- Before Selection

  Parents

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6	7	1	2	5	3	0	4

  Children

1	7	6	5	2	3	4	0

6	1	3	2	7	5	0	4

Selection Process

- Compute fitness of parents and children

  Parents fitness

1.234	0.937

  Children fitness

1.895	0.845

Selection Process

- Find worst and best

  Parents fitness

1.234	0.937

  Children fitness

1.895	0.845

Selection Process

- After Selection

  New Parents

7	1	6	3	2	5	4	0

1	7	6	5	2	3	4	0

Mutation Process

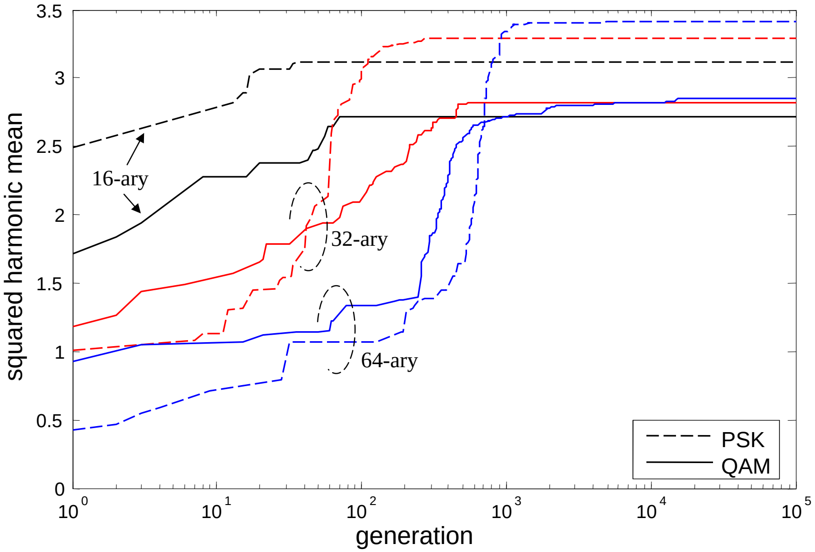
- Parents fitness

  1.234 0.937

  Children fitness

  0.340 0.243
- Do mutation by swapping two positions
  - | 1 | 7 | 6 | 5 | 2 | 3 | 4 | 0 | ⟹ | 1 | 7 | 4 | 5 | 2 | 3 | 6 | 0 |

Mapping Optimization Results



Comparing RTS and GA

M	Modulation	d_h^2 from RTS	d_h^2 from GA	generations
16	QAM	2.7190	2.7190	103
	PSK	3.1142	3.1142	1,249
32	QAM	2.8154	2.8154	542
	PSK	3.2916	3.2916	530
64	QAM	2.8742	2.8460	15,421
	PSK	3.4102	3.4102	8,987

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2. Convergence speed of GA is similar to RTS

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Constellation Optimization

- Mapping optimization is not the only way to get low error floors
- For a particular mapping following function is to be optimized

$$\min_{\mathcal{X},\boldsymbol{\mu}} \sum_{k=0}^{m-1} \sum_{x' \in \mathcal{X}_k^{(1)}} ||x' - g_k(x')||^{-2}. \tag{6}$$

- Constellation optimization search space is infinite and is very challenging
- A heuristic method was developed to generate a constellation that has low error rate.

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Technique for constellation optimization

1. Choose PSK constellation as it has best d_h^2
2. Optimize mapping for constellation in an attempt to increase d_h^2
3. Pick an EFF pair that has minimum d_e, defined by

$$d_e = \min_{\substack{x' \in \mathcal{X}_k^{(1)} \\ 0 \le k \le m-1}} ||x' - g_k(x')||. \tag{7}$$

4. These two points are then forced to be at distance αd_e, where $\alpha \ge 1$, arbitrarily chosen as 1.01
5. Renormalize the constellation
6. Above process is repeated iteratively from step 2

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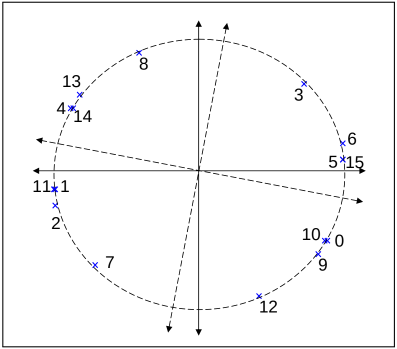
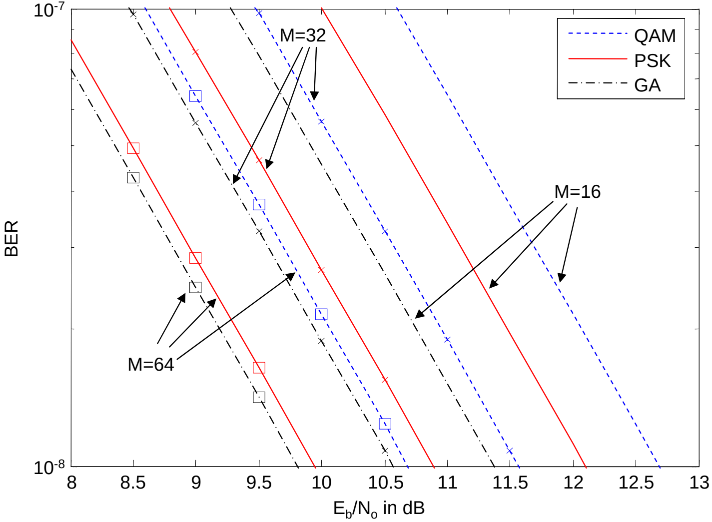
Optimized 16-ary constellation

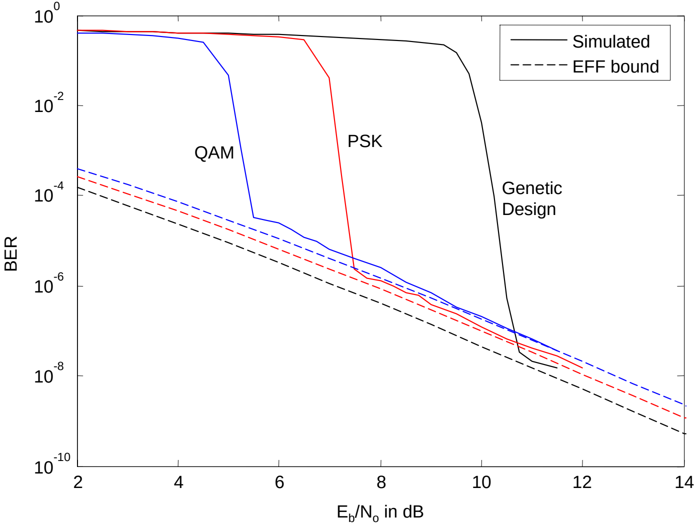
M	Modulation	d_h^2
16	QAM	2.718954
	PSK	3.114162
	GA	3.684133
32	QAM	2.815367
	PSK	3.291638
	GA	3.547659
64	QAM	2.874222
	PSK	3.410212
	GA	3.517697



Improved EFF Bound on BER using (7,5) Convolutional Code



Simulated BER using 16-ary modulation



Conclusion

- Genetic Algorithm can be used to optimize mapping
- New constellation can be evolved for BICM-ID with low error floors by optimizing the placement of constellation points
- Low error floors for new constellation comes at cost of shifting water-fall region to high SNR

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