The Information-Outage Probability of Finite-Length Codes over AWGN Channels

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Outline

1. A Review of Mutual Information
2. Information-Outage Probability
3. Bounding the Achievable Error Probability
4. Conclusion

Motivation

- Shannon Channel Capacity
  - Gives reliable rate for communication at specific SNR.
  - Requires infinite-length codewords.
  - In practice, codes are restricted to finite lengths.
- Predicting Performance of Finite-Length Codes
  - Several bounds based on blocklength exist, but are non-trivial to calculate.
  - Want information-theoretic metric which is a function of blocklength.
  - Information-outage probability used previously with block fading channels.
  - Apply information-outage probability based on the mutual information rate.
  - Compare with coded performance and alternative bounds.

Mutual Information

- Let X be input complex Gaussian process to an AWGN channel

$$Y = X + N$$

where N is complex AWGN process, and Y is channel output.

- If x and y are samples of the processes X and Y, we can write the *mutual information* between the random variables, x and y as

$$i(x;y) = \log \frac{f_{X,Y}(x,y)}{f_X(x) f_Y(y)}.$$

- The *average mutual information* is the expectation of the mutual information random variable

$$I(X;Y) = E\left[i(X;Y)\right]$$

Mutual Information of AWGN Channel

- Mutual information can also be expressed as

$$i(x;y) \quad = \quad \log \frac{f_{Y|X}(y|x)}{f_Y(y)}$$

- Both of these distributions are known.
- Substituting pdf for $Y|X$ and Y

$$i(x;y) \quad = \quad \log\left(1 + \frac{\mathcal{E}_s}{N_o}\right) + \frac{|y|^2}{\mathcal{E}_s + N_0} - \frac{|y-x|^2}{N_0}$$

Relationship between Capacity and MI

- We can find the *average mutual information* by taking the expectation of the mutual information random variable

$$
\begin{aligned}
I(X;Y) &= E\left[i(X;Y)\right] \\
&= \log\left(1+\frac{\mathcal{E}_s}{N_0}\right) + \frac{E\left[|y|^2\right]}{\mathcal{E}_s+N_0} - \frac{E\left[|y-x|^2\right]}{N_0} \\
&= \log\left(1+\frac{\mathcal{E}_s}{N_0}\right)
\end{aligned}
$$

- This is also known as the *ergodic capacity* of the channel.
  - Assumption that codeword length goes to infinity.
  - What if this is not the case?

Mutual Information Rate

- Let $\mathbf{x}$ be a vector of n samples of the input process X
- Mutual information between input sample vector $\mathbf{x}$ and output sample vector $\mathbf{y}$ is

$$\begin{aligned} i(\mathbf{x};\mathbf{y}) &= \frac{1}{n}\log\frac{f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})}{f_{\mathbf{X}}(\mathbf{x})f_{\mathbf{Y}}(\mathbf{y})} \\ &= \frac{1}{n}\sum_{k=1}^{n} i\left(x_k;y_k\right) \end{aligned}$$

- Also known as *mutual information rate*.
- Expressions equivalent due to i.i.d. channel inputs and white noise.
- We know mutual information for single input sample; now need to find average of n samples.

Distribution of Mutual Information

- Let Z_n be the mutual information rate between the channel input and output vectors, $\mathbf{x}$ and $\mathbf{y}$

$$Z_n = \log\left(1 + \frac{\mathcal{E}_s}{N_0}\right) + W_n$$

where W_n is the average of n i.i.d. Laplacian random variables [Laneman, 2006], each with zero mean and variance

$$\sigma_W^2 = \frac{2\mathcal{E}_s}{\mathcal{E}_s + N_0}.$$

- W_n is a Bessel-K random variable with pdf given by:

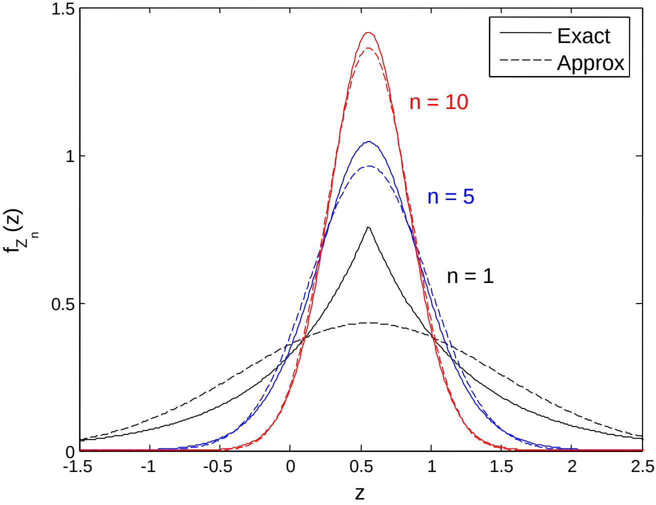
$$f_{W_n}(w) = \frac{2^{1-n}}{\sqrt{\pi}\Gamma(n)\sigma_W}\left(\frac{\sqrt{2}|w|}{\sigma_W}\right)^{n-\frac{1}{2}} K_{n-\frac{1}{2}}\left(\frac{\sqrt{2}|w|}{\sigma_W}\right)$$

Gaussian Approximation

- The Central Limit Theorem tells us that the sum of n i.i.d. random variables with finite mean and variance will approach a Gaussian distribution as $n \to \infty$
- Exact Bessel-K distribution will approach a Gaussian distribution as blocklength increases.
- We introduce a Gaussian approximation to the mutual information rate.

$$\tilde{Z}_n \sim \mathcal{N}\left(\log\left(1 + \frac{\mathcal{E}_s}{N_0}\right), \frac{2\mathcal{E}_s}{n\left(\mathcal{E}_s + N_0\right)}\right)$$

Distribution Comparison



Defining Information-Outage Probability

- Let $R_2 = k/n$ represent the code rate in bits per symbol.
  - $R_e = \log(2) R_2$ is the equivalent rate in nats per symbol.
- Blocklength, n, is finite, so mutual information rate is random.
- An *outage* occurs when the mutual information rate is less than the code rate.
- Therefore the *outage probability* is defined as

$$\begin{aligned} P_o &= P\left[Z_n \le R_e\right] = F_{Z_n}\left(R_e\right) \\ &= P\left[\log\left(1 + \frac{\mathcal{E}_s}{N_0}\right) + W_n \le R_e\right] \\ &= F_{W_n}\left(R_e - \log\left(1 + \frac{\mathcal{E}_s}{N_0}\right)\right) \end{aligned}$$

- We need the CDF of the random variable W_n to calculate the information-outage probability.

Exact and Approximate CDFs

- Recall that W_n is a Bessel-K random variable, which has CDF

$$F_{W_n}(w) = 1 - \sum_{l=0}^{n-1} \frac{\Gamma(n+l)\Gamma(n-l, \sqrt{2}w/\sigma)}{\Gamma(n)\Gamma(n-l)\Gamma(l+1)} 2^{-n-l}$$

  for $w \geq 0$ and

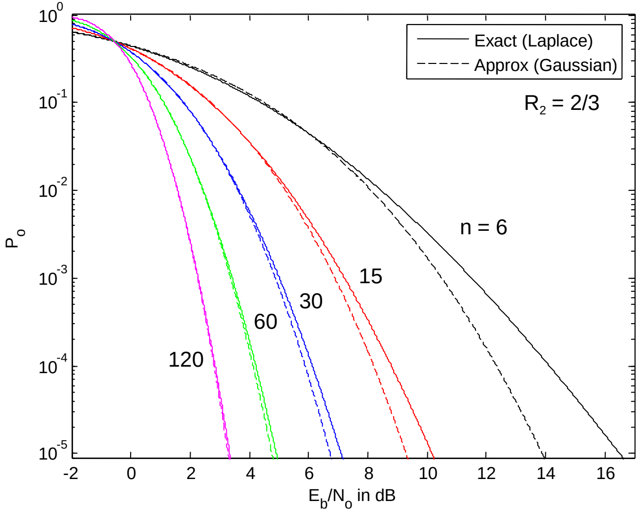
$$F_{W_n}(w) = \sum_{l=0}^{n-1} \frac{\Gamma(n+l)\Gamma(n-l, -\sqrt{2}w/\sigma)}{\Gamma(n)\Gamma(n-l)\Gamma(l+1)} 2^{-n-l}$$

  for $w < 0$.

- Alternatively, the CDF of the Gaussian approximation can be found by using the Q-function

$$F_{\tilde{Z}_n}(z) = Q\left(\frac{\log\left(1 + \frac{\mathcal{E}_s}{N_0}\right) - z}{\sqrt{\frac{2\mathcal{E}_s}{n(\mathcal{E}_s + N_0)}}} \right).$$

IOP Comparison

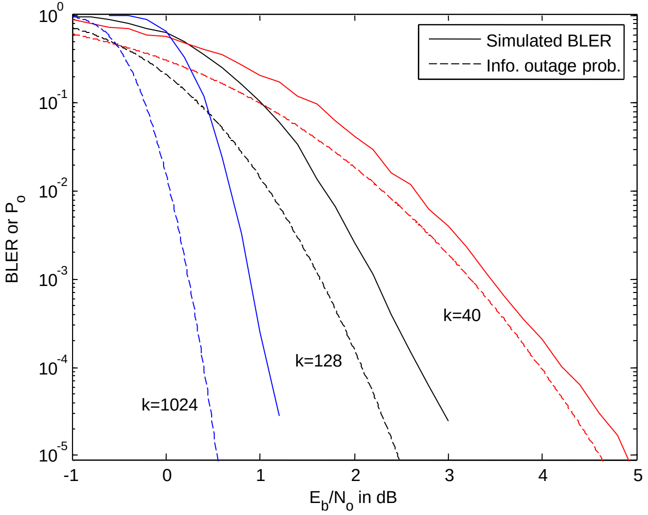


Turbo Code Performance

- Information-outage probability as a predictor of performance.
  - How does IOP compare to an existing capacity-approaching code?
- UMTS LTE (long term evolution) turbo code
  - Supports 188 distinct values of information block size, k, in bits.
  - Codeword blocklength defined as $n = 3k + 12$.
  - Simulated with QPSK, binary rate given by

$$R_2 = 2\frac{k}{n} = \frac{2k}{3k + 12} \approx 2/3$$

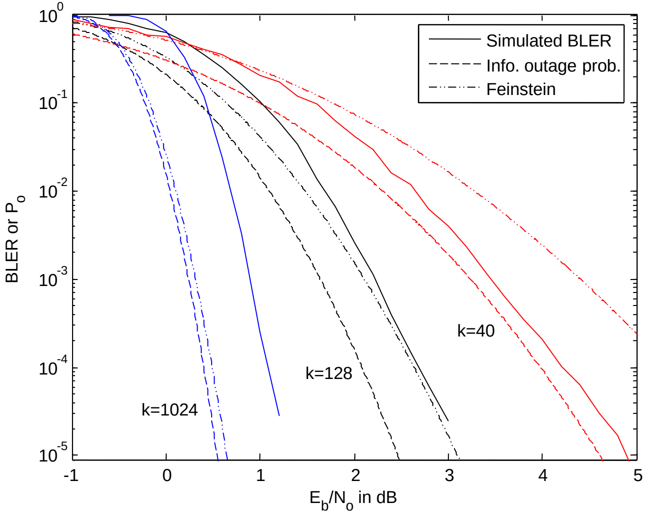
IOP vs. LTE


Simulated BLER
Info. outage prob.
k=40
k=128
k=1024
BLER or P_o
E_b/N_o in dB
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
-1
0
1
2
3
4
5

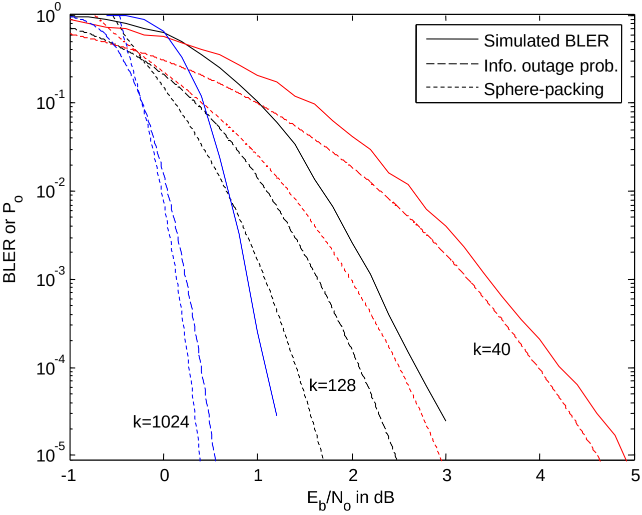
Alternative Bounds

- Feinstein's Lemma, [Feinstein, 1954].
  - Bound on maximal codeword error rate, based on mutual information rate.
  - States that a code exists that can achieve a specific codeword error probability.
  - Codes may exist that perform better than bound.
- Sphere-Packing Bound, [Shannon, 1959].
  - Lower bound on codeword-error probability based on n-dimensional Euclidian space.
  - Sphere in n-dimensional space is packed with $M = 2^k$ cones.
- Random Coding Bound, [Shannon, 1959].
  - Bound on the ensemble average word-error probability.
  - Averaged over all possible (n, k) codes from randomly selected set of 2^k codewords.

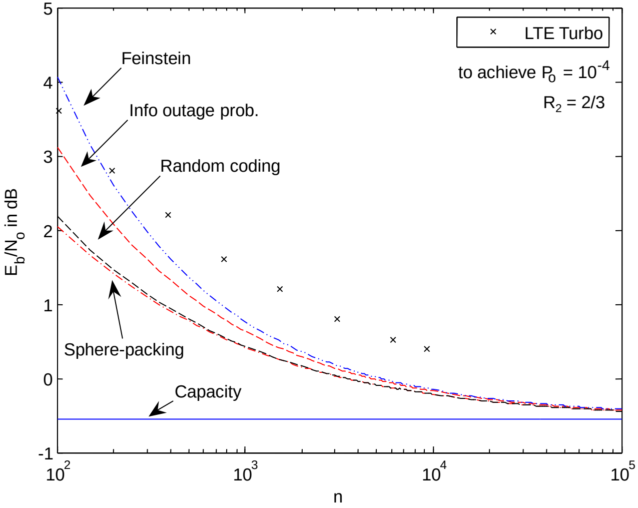
IOP vs. Feinstein



IOP vs. Sphere-Packing Bound



Bound Comparison



Conclusion

- Distribution of Mutual Information Rate
  - Exact: Mean-shifted Bessel-K distribution.
  - Approximation: Gaussian distribution.
  - As blocklength increases,
    - Exact distribution becomes increasingly difficult to calculate as numerical stability becomes a factor.
    - However, exact distribution approaches Gaussian approximation.
- Information-Outage Probability
  - Useful predictor of error performance.
  - Calculation using Gaussian approximation is trivial compared to other previously derived bounds.

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Thank You.