# Analysis and Optimization of a Frequency-Hopping Ad Hoc Network in Rayleigh Fading

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# Outline

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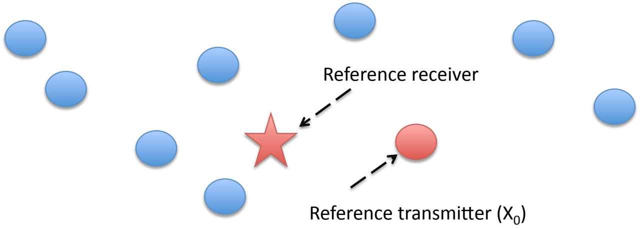
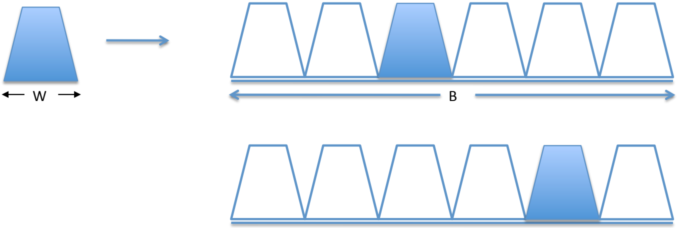
# Ad Hoc Networks




- Mobile transmitters are randomly placed in 2-D space.
- A reference receiver is located at the origin.
- $X_i$ represents the $i^{th}$ transmitter and its location.
  - $X_0$ is location of the reference transmitter.
  - $M$ interfering transmitters, $\{X_1, ..., X_M\}$.
  - $|X_i|$ is distance from $i^{th}$ transmitter to the reference receiver.
- Spatial models
  - Interferers placed independently and uniformly over the network area.
  - $M$ can be fixed (BPP) or variable (PPP).

# Frequency Hopping

- To manage interference, *frequency hopping* (FH) is used:




- Each mobile transmits with *duty factor* $d$.
  - Likelihood of a transmission is $d \leq 1$.
- Each transmitting mobile randomly picks from among $L$ frequencies.
  - Probability of a *collision* is $p = d/L = 1/L'$, where $L' = L/d$.

# SINR

The performance at the reference receiver is characterized by the *signal-to-interference and noise ratio* (SINR), given by:

$$\gamma = \frac{g_0 \Omega_0}{\Gamma^{-1} + \sum_{i=1}^{M} I_i g_i \Omega_i} \quad (1)$$

where:

- $\Gamma$ is the SNR at unit distance.
- $g_i$ is the power gain due to Rayleigh fading (an exponential variable).
- $I_i$ is a Bernoulli variable that indicates a collision.
  - $P\left[I_i = 1\right] = p = 1/L'$.
- $\Omega_i = \frac{P_i}{P_0}||X_i||^{-\alpha}$ is the normalized receiver power from transmitter $i$.
  - $P_i$ is the power of transmitter $i$, assumed to be constant for all $i$.
  - $\alpha$ is the path loss.
  - $||X_0|| = 1$ to normalize distance.

# Outline

# Outage Probability

- An *outage* occurs when the SINR is below a threshold $\beta$.
    - $\beta$ depends on the choice of modulation and coding.
- The *outage probability* is

$$\epsilon = P\Big[\gamma \leq \beta\Big]. \tag{2}$$

- Substituting (1) into (2) and rearranging yields

$$\epsilon = P\Big[\underbrace{\beta^{-1} g_0 \Omega_0 - \sum_{i=1}^{M} I_i g_i \Omega_i}_{\mathsf{Z}} \leq \Gamma^{-1}\Big].$$

- The outage probability is related to the cdf of Z,

$$\epsilon = P\left[\mathsf{Z} \leq \Gamma^{-1}\right] = F_{\mathsf{Z}}(\Gamma^{-1}).$$

# Outage Probability Derivation

The cdf conditioned
on the network geometry, $\mathbf{\Omega}$:
$$\epsilon_\Omega = P\left[\gamma \leq \beta \middle| \mathbf{\Omega}\right] = F_{\mathsf{Z}}(\Gamma^{-1} \middle| \mathbf{\Omega})$$

# Outage Probability Derivation

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The cdf conditioned
on the number of interferers, $M$:
$\epsilon_M = \mathbb{E}\left[\epsilon_\Omega\right] =$
$F_{\mathsf{Z}_M}(\Gamma^{-1}) = \int f_{\mathbf{\Omega}}(\boldsymbol{\omega}) F_{\mathsf{Z}}(\Gamma^{-1} \middle| \boldsymbol{\omega}) d\boldsymbol{\omega}$

$$f_{\mathbf{\Omega}}(\boldsymbol{\omega}) = \prod_{i=1}^{M} f_{\Omega_i}(\omega_i)$$
is the pdf of $\mathbf{\Omega}$

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$$f_{\mathbf{\Omega}}(\boldsymbol{\omega}) = \prod_{i=1}^{M} f_{\Omega_i}(\omega_i)$$

is the pdf of $\mathbf{\Omega}$

The unconditional cdf:

$$\epsilon = \mathbb{E}\left[\epsilon_M\right] =$$

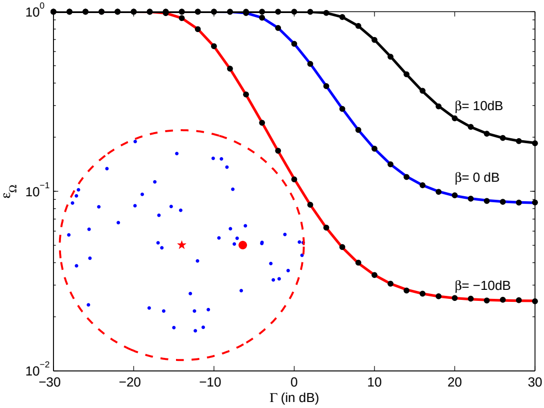
$$F_{\mathsf{Z}}(\Gamma^{-1}) = \sum_{m=0}^{\infty} p_M(m) F_{\mathsf{Z}_m}(z)$$

$p_M(m)$
is the pmf of M

# Conditional Outage Probability

- The outage probability conditioned on the network geometry:

$$F_Z(z|\mathbf{\Omega}) = 1 - e^{-\beta\Omega_0^{-1}z}\prod_{i=1}^{M}\left[\frac{(1-p_i)\beta\Omega_0^{-1}+\Omega_i^{-1}}{\beta\Omega_0^{-1}+\Omega_i^{-1}}\right] \quad (3)$$




Example:

- $M = 50$ interferers.
- Annular network.
- $r_{ex} = 0.25$ inner radius.
- $r_{net} = 2$ outer radius.
- $\alpha = 3$.
- $L' = 200$.
- Analytical curves are solid, while • represents simulated values.

# Average Outage Probability for a BPP: Results

- In a binomial point process (BPP), a fixed number of interferers are independently placed according to a uniform distribution.
- The cdf with a BPP network is:

$$F_{\mathsf{Z}_M}(z) = \int f_{\boldsymbol{\Omega}}(\boldsymbol{\omega}) F_{\mathsf{Z}}(z \big| \boldsymbol{\omega}) d\boldsymbol{\omega} \quad (4)$$

where

$$f_{\Omega_i}(\omega) = \frac{2\omega^{\frac{2-\alpha}{\alpha}}}{\alpha\left(r_{net}^2 - r_{ex}^2\right)} \quad \text{for } r_{ex}^\alpha \leq \omega \leq r_{net}^\alpha.$$

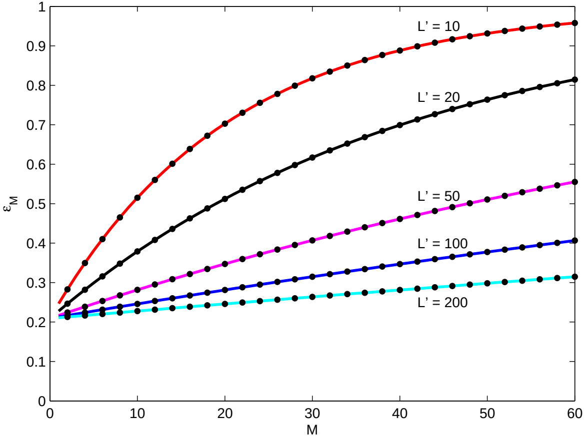
Substituting (3) into (4), the cdf of $\mathsf{Z}_M$ is

$$F_{\mathsf{Z}_M}(z) = 1 - e^{-\beta\Omega_0^{-1}z}\left[\frac{\Psi\left(r_{net}^\alpha\right) - \Psi\left(r_{ex}^\alpha\right)}{r_{net}^2 - r_{ex}^2}\right]^M \quad (5)$$

where

- $\Psi(x) = x^{\frac{2}{\alpha}}\left[1 - p + \frac{2p}{\alpha+2} \cdot \frac{x^{\frac{2+\alpha}{\alpha}}}{\beta\Omega_0^{-1}} \cdot {}_2F_1\left(\left[1, \frac{\alpha+2}{\alpha}\right]; \frac{2\alpha+2}{\alpha}, -\frac{x}{\beta\Omega_0^{-1}}\right)\right].$

# Average Outage Probability for a BPP: Example




Example:

- $r_{ex} = 0.25.$
- $r_{net} = 2.$
- $\alpha = 3.$
- $\beta = 3.7$ dB.
- $\Gamma = 10$ dB.

Figure: Outage probability $\epsilon_M$ as a function of $M$ for five values of $L'$ when the location of the nodes is drawn from a BPP. Analytical curves are solid, while • represents simulated values.

# Average Outage Probability for a PPP: Results

- In a Poisson point process (PPP), the number of interferers $M$ is a Poisson variable.
- The cdf with a PPP network is:

$$F_{\mathsf{Z}}(z) = \sum_{m=0}^{\infty} p_M(m) F_{\mathsf{Z}_m}(z) = \sum_{m=0}^{\infty} \frac{(\lambda A)^m}{m!} e^{-\lambda A} F_{\mathsf{Z}_m}(z) \quad (6)$$

where
  - $\lambda$ is the density of the interferers per unit area;
  - $A = \pi(r_{net}^2 - r_{ex}^2)$ is the area of the network;

substituting (5) into (6), the average outage probability is:

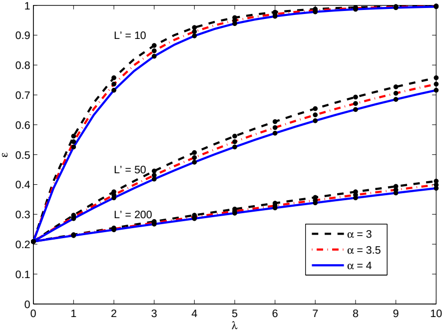
$$F_{\mathsf{Z}}(z) = 1 - e^{-\beta\Omega_0^{-1} z} e^{-\pi\lambda \cdot \left\{ r_{net}^2 - r_{ex}^2 - [\Psi(r_{net}^\alpha) - \Psi(r_{ex}^\alpha)] \right\}} \quad (7)$$

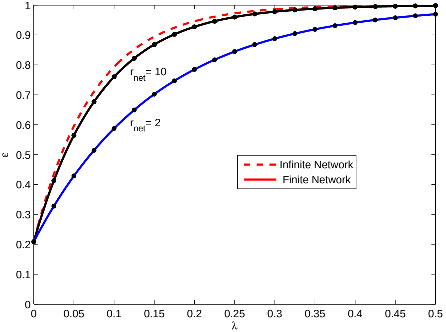
- When $r_{net} \to \infty$, $r_{ex} = 0$, and $p = L' = 1$:

$$F_{\mathsf{Z}}(z) = 1 - e^{-\beta_0 z} e^{\frac{-2\pi\lambda}{\alpha} \beta_0^{\frac{2}{\alpha}} \Gamma\left(\frac{2}{\alpha}\right) \Gamma\left(1 - \frac{2}{\alpha}\right)}$$

  - The same expression is obtained in literature by Baccelli et al. [4].

# Average Outage Probability for a PPP: Examples




Parameters:

- $r_{ex} = 0.25$;
- $r_{net} = 2$;
- $\beta = 3.7$ dB;
- $\Gamma = 10$ dB.

Parameters:

- $r_{ex} = 0$;
- $\alpha = 3$;
- $\beta = 3.7$ dB;
- $\Gamma = 10$ dB.
- $L' = 1$;

# Outline

# Transmission Capacity: Definition

- The *transmission capacity* is the spatial spectral efficiency.
- If there are $\lambda$ mobiles per unit area, then the number of successful transmissions per unit area is

$$\tau = \lambda(1-\epsilon)$$

- If the outage probability $\epsilon$ is constrained to not exceed $\zeta$, then the transmission capacity is

$$\tau_c(\zeta) = \epsilon^{-1}(\zeta)(1-\zeta) \tag{8}$$

where $\epsilon^{-1}(\zeta)$ is the maximum mobile density such that $\epsilon \leq \zeta$.

# Transmission Capacity: Results

- For the BPP case, $\epsilon^{-1}(\zeta)$ is found be solving $\epsilon = F_{\mathsf{Z}_M}(\Gamma^{-1}) = \zeta$ for $\lambda = M/A$ and then substituting into (8):

$$\tau_c(\zeta) = \frac{(1-\zeta)\left[\log(1-\zeta) + \beta\Omega_0^{-1}\Gamma^{-1}\right]}{A\log\left\{\left(r_{net}^2 - r_{ex}^2\right)^{-1}\left[\Psi\left(r_{net}^{\alpha}\right) - \Psi\left(r_{ex}^{\alpha}\right)\right]\right\}}$$

- In the PPP case, $\epsilon^{-1}(\zeta)$ is found be solving $\epsilon = F_{\mathsf{Z}}(\Gamma^{-1}) = \zeta$ for $\lambda$ and then substituting into (8):

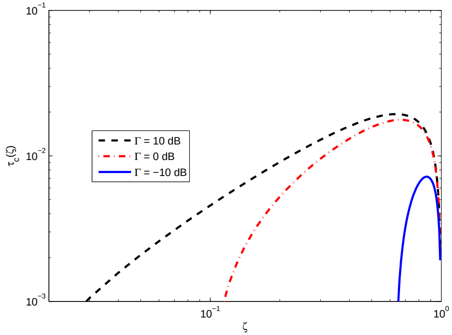
$$\tau_c(\zeta) = \frac{(1-\zeta)\left[\log(1-\zeta)^{-1} - \beta\Omega_0^{-1}\Gamma^{-1}\right]}{\pi\left\{r_{net}^2 - r_{ex}^2 - \left[\Psi\left(r_{net}^{\alpha}\right) - \Psi\left(r_{ex}^{\alpha}\right)\right]\right\}}.$$

- Asymptotically for a PPP, with $r_{ex} = 0$ and $p = 1$, as $r_{net} \to \infty$:

$$\tau_c(\zeta) = \frac{(1-\zeta)\left[\log(1-\zeta)^{-1} - \beta\Omega_0^{-1}\Gamma^{-1}\right]}{\pi\beta_0^{\frac{2}{\alpha}}\frac{2\pi}{\alpha}\csc\left(\frac{2\pi}{\alpha}\right)}$$

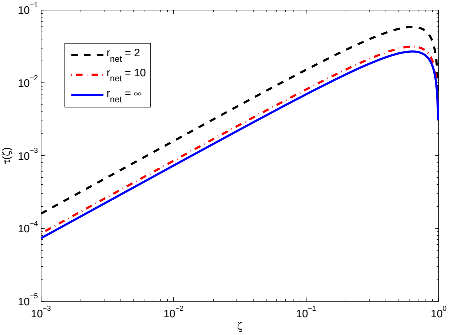
  - The same expression is obtained in literature by Weber et al. [14].

# Transmission Capacity: Examples




Parameters:

- $r_{ex} = 0$;
- $r_{net} = 2$;
- $L' = 1$.
- $\beta = -10$ dB;
- $\alpha = 3$;




Parameters:

- $r_{ex} = 0$;
- $\Gamma = 10$ dB.
- $L' = 1$;
- $\beta = -10$ dB;
- $\alpha = 3$;

# Outline

# Accounting for Modulation

- Until now, we have picked the SINR threshold $\beta$ arbitrarily.
- $\beta$ depends on the choice of modulation.
  - For *ideal* (Gaussian-input) signaling

$$C(\gamma) = \log_2(1+\gamma)$$

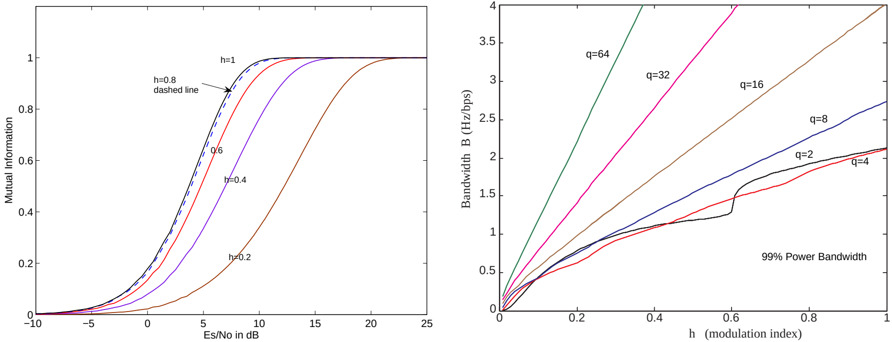
    $\beta$ is the value of $\gamma$ for which $C(\gamma) = R$ (the code rate),

$$\beta = 2^R - 1$$

  - For other modulations, the *modulation-constrained* capacity $C(\gamma)$ must be used, which can be found by measuring the mutual information between channel input and output.
- Additionally, the code rate $R$ and the spectral-efficiency of the modulation $\eta$ can be taken into account to give transmission capacity in units of bps/Hz/$m^2$.

# Noncoherent Binary CPFSK

FH systems often use noncoherent continuous-phase frequency-shift keying, whose capacity & bandwidth depends on the modulation index $h$.




(a) channel capacity versus $\mathcal{E}_S/N_0$ (b) bandwidth versus modulation index

[9] S. Cheng, R. Iyer Sehshadri, M.C. Valenti, and D. Torrieri, "The capacity of noncoherent continuous-phase frequency shift keying", in *Proc. Conf. on Info. Sci. and Sys. (CISS)*, (Baltimore, MD), Mar. 2007.

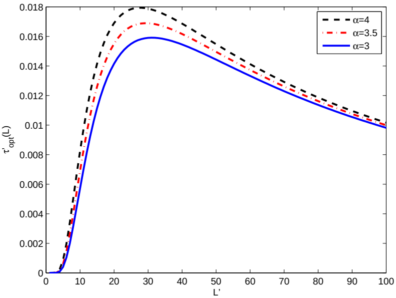
# Modulation-Constrained Transmission Capacity

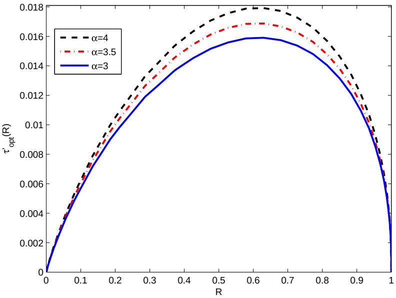
- The *modulation-constrained* transmission capacity is

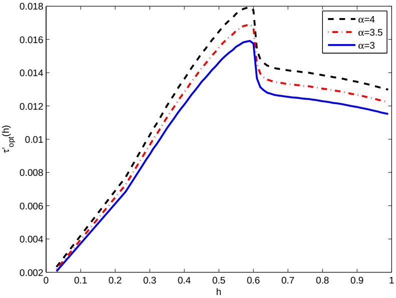
$$\tau' = \lambda(1-\epsilon)\left(\frac{R\eta(h)}{L'}\right)$$

  where
  - $R$ is the rate of the channel code.
  - $h$ is the modulation index.
  - $\eta(h)$ the modulation's spectral efficiency (bps/Hz).
  - $\epsilon = P[C(\gamma) \leq R] = P[\gamma \leq C^{-1}(R)]$.
  - $L'$ is the effective number of hopping channels.
  - $\lambda$ is the density of interferers.
  - $\tau'$ has units of $bps/Hz/m^2$.
- For a given $\lambda$, $\Gamma$, and spatial model, there is a set of $(L', R, h)$ that maximizes $\tau'$.

# Influence of Parameters: BPP



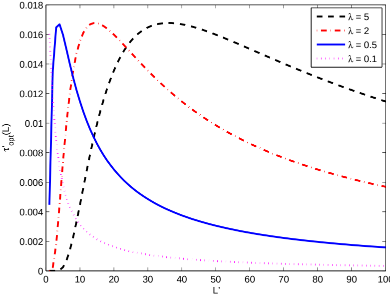



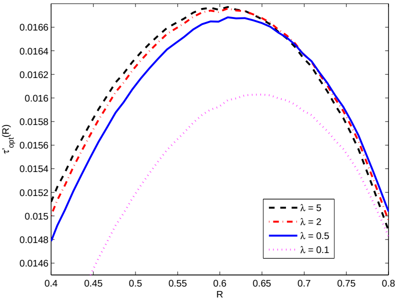



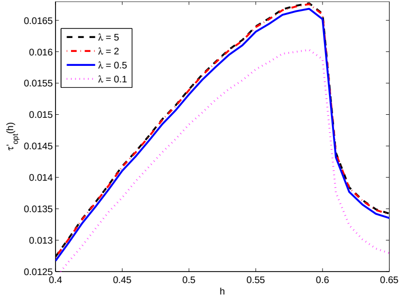

Parameters:

- $r_{ex} = 0.25$;
- $r_{net} = 2$;
- $\Gamma = 10$ dB.
- $M = 50$.

# Influence of Parameters: PPP




Parameters:

- $r_{ex} = 0.25;$
- $r_{net} = 2;$
- $\Gamma = 10$ dB.
- $\alpha = 3.$

# Outline

# Optimization Objectives

- For a given network model $(r_{ex}, r_{net}, \alpha, \lambda)$ and channel SNR $\Gamma$, the values of $(L', R, h)$ that maximize the modulation-constrained transmission capacity $\tau'$ are found.
- Optimization is through exhaustive search or a gradient-search strategy.
- Performed for both BPP and PPP spatial models.

# Example of optimization for a BPP

| $r_{net}$ | $r_{ex}$ | $\alpha$ | $L'$ | $R$ | $h$ | $\tau'_{opt}$ |
|---|---|---|---|---|---|---|
| 2 | 0.25 | 3 | 32 | 0.62 | 0.59 | 0.01590 |
| | | 3.5 | 30 | 0.62 | 0.59 | 0.01688 |
| | | 4 | 28 | 0.62 | 0.59 | 0.01792 |
| | 0.5 | 3 | 30 | 0.61 | 0.59 | 0.01641 |
| | | 3.5 | 29 | 0.61 | 0.59 | 0.01752 |
| | | 4 | 26 | 0.59 | 0.59 | 0.01871 |
| 4 | 0.25 | 3 | 12 | 0.54 | 0.59 | 0.00983 |
| | | 3.5 | 10 | 0.55 | 0.59 | 0.01187 |
| | | 4 | 8 | 0.54 | 0.59 | 0.01395 |
| | 0.5 | 3 | 11 | 0.52 | 0.59 | 0.01024 |
| | | 3.5 | 9 | 0.52 | 0.59 | 0.01252 |
| | | 4 | 8 | 0.55 | 0.59 | 0.01484 |

Table: *Results of the Optimization for an annular network area where the interferers are drawn from a BPP. The number of interferers is fixed to $M = 50$.*

# Example of optimization for a PPP

| $r_{net}$ | $r_{ex}$ | $\alpha$ | $L'$ | $R$ | $h$ | $\tau'_{opt}$ |
|---|---|---|---|---|---|---|
| 2 | 0.25 | 3 | 8 | 0.62 | 0.59 | 0.01597 |
| | | 3.5 | 7 | 0.62 | 0.59 | 0.01697 |
| | | 4 | 7 | 0.62 | 0.59 | 0.01801 |
| | 0.5 | 3 | 7 | 0.61 | 0.59 | 0.01731 |
| | | 3.5 | 7 | 0.61 | 0.59 | 0.01845 |
| | | 4 | 6 | 0.59 | 0.59 | 0.01973 |
| 4 | 0.25 | 3 | 12 | 0.52 | 0.59 | 0.00977 |
| | | 3.5 | 10 | 0.54 | 0.59 | 0.01180 |
| | | 4 | 8 | 0.54 | 0.59 | 0.01387 |
| | 0.5 | 3 | 11 | 0.52 | 0.59 | 0.01030 |
| | | 3.5 | 9 | 0.53 | 0.59 | 0.01258 |
| | | 4 | 8 | 0.55 | 0.59 | 0.01491 |

Table: *Results of the Optimization for an annular network area where the interferers are drawn from a PPP. The intensity $\lambda$ per unit area is fixed to $\lambda = 1$.*

# Outline

# Conclusions

- The performance of frequency-hopping ad hoc networks is a function of many parameters.
  - Number of hopping channels $L$.
  - Code rate $R$.
  - Modulation index $h$ (if CPFSK modulation).
- These parameters can be jointly optimized.
  - Transmission capacity is the objective function of choice.
  - The modulation-constrained TC quantifies the tradeoffs involved.
- The approach is general enough to handle a wide variety of conditions
  - Can be extended to accommodate Nakagami fading and shadowing.
  - Any spatial model, including repulsion models.
  - Adjacent-channel interference due to spectral splatter.
  - Adaptive code rates ($R$ not fixed for all users).
- Additional constraints can be imposed on the optimization.
  - Per-node outage constraint.
  - Fixed or minimum data rates per user.

# Thank You