Analysis and Optimization of a Frequency-Hopping Ad Hoc Network in Rayleigh Fading

S. Talarico $^1$ M. C. Valenti $^1$ D. Torrieri $^2$

$^1$West Virginia University
Morgantown, WV

$^2$U.S. Army Research Laboratory
Adelphi, MD

May 30, 2012
1. Frequency-Hopping Ad Hoc Networks
2. Outage Probability
3. Transmission Capacity
4. Modulation Constraints
5. Optimization Results
6. Conclusions
Outline

1. Frequency-Hopping Ad Hoc Networks
2. Outage Probability
3. Transmission Capacity
4. Modulation Constraints
5. Optimization Results
6. Conclusions
Mobile transmitters are randomly placed in 2-D space.

A reference receiver is located at the origin.

$X_i$ represents the $i^{th}$ transmitter and its location.

- $X_0$ is location of the reference transmitter.
- $M$ interfering transmitters, $\{X_1, ..., X_M\}$.
- $|X_i|$ is distance from $i^{th}$ transmitter to the reference receiver.

Spatial models

- Interferers placed independently and uniformly over the network area.
- $M$ can be fixed (BPP) or variable (PPP).
To manage interference, frequency hopping (FH) is used:

- Each mobile transmits with duty factor $d$.
- Likelihood of a transmission is $d \leq 1$.
- Each transmitting mobile randomly picks from among $L$ frequencies.
- Probability of a collision is $p = d/L = 1/L'$, where $L' = L/d$. 
The performance at the reference receiver is characterized by the signal-to-interference and noise ratio (SINR), given by:

$$\gamma = \frac{g_0 \Omega_0}{\Gamma^{-1} + \sum_{i=1}^{M} I_i g_i \Omega_i} \tag{1}$$

where:

- $\Gamma$ is the SNR at unit distance.
- $g_i$ is the power gain due to Rayleigh fading (an exponential variable).
- $I_i$ is a Bernoulli variable that indicates a collision.
  - $P[I_i = 1] = p = 1/L'$.
- $\Omega_i = \frac{P_i}{P_0} ||X_i||^{-\alpha}$ is the normalized receiver power from transmitter $i$.
- $P_i$ is the power of transmitter $i$, assumed to be constant for all $i$.
- $\alpha$ is the path loss.
- $||X_0|| = 1$ to normalize distance.
Outline

1. Frequency-Hopping Ad Hoc Networks
2. Outage Probability
3. Transmission Capacity
4. Modulation Constraints
5. Optimization Results
6. Conclusions
An outage occurs when the SINR is below a threshold $\beta$. $\beta$ depends on the choice of modulation and coding.

The outage probability is

$$\epsilon = P[\gamma \leq \beta].$$

(2)

Substituting (1) into (2) and rearranging yields

$$\epsilon = P \left[ \beta^{-1} g_0 \Omega_0 - \sum_{i=1}^{M} I_i g_i \Omega_i \leq \Gamma^{-1} \right].$$

The outage probability is related to the cdf of $Z$,

$$\epsilon = P \left[ Z \leq \Gamma^{-1} \right] = F_Z(\Gamma^{-1}).$$
Outage Probability Derivation

The cdf conditioned on the network geometry, $\Omega$:

$$\epsilon_\Omega = P[\gamma \leq \beta | \Omega] = F_Z(\Gamma^{-1} | \Omega)$$
The cdf conditioned on the network geometry, $\Omega$:

$$\epsilon_{\Omega} = P[\gamma \leq \beta \mid \Omega] = F_Z(\Gamma^{-1} \mid \Omega)$$

The cdf conditioned on the number of interferers, $M$:

$$\epsilon_M = \mathbb{E}[\epsilon_{\Omega}] = F_{Z_M}(\Gamma^{-1}) = \int f_{\Omega}(\omega) F_Z(\Gamma^{-1} \mid \omega) d\omega$$

$$f_{\Omega}(\omega) = \prod_{i=1}^{M} f_{\Omega_i}(\omega_i)$$

is the pdf of $\Omega$. 
Outage Probability Derivation

The cdf conditioned on the network geometry, $\Omega$:
$$\epsilon_\Omega = P \left[ \gamma \leq \beta \mid \Omega \right] = F_Z(\Gamma^{-1} \mid \Omega)$$

The cdf conditioned on the number of interferers, $M$:
$$\epsilon_M = \mathbb{E} [\epsilon_\Omega] = F_{Z_M}(\Gamma^{-1}) = \int f_\Omega(\omega) F_Z(\Gamma^{-1} \mid \omega) d\omega$$

The unconditional cdf:
$$\epsilon = \mathbb{E} [\epsilon_M] = F_Z(\Gamma^{-1}) = \sum_{m=0}^{\infty} p_M(m) F_{Z^m}(z)$$

$f_\Omega(\omega) = \prod_{i=1}^{M} f_{\Omega_i}(\omega_i)$ is the pdf of $\Omega$

$p_M(m)$ is the pmf of $M$
Conditional Outage Probability

The outage probability conditioned on the network geometry:

$$F_Z(z | \Omega) = 1 - e^{-\beta \Omega_0^{-1} z} \prod_{i=1}^{M} \left[ \frac{(1 - p_i) \beta \Omega_0^{-1} + \Omega_i^{-1}}{\beta \Omega_0^{-1} + \Omega_i^{-1}} \right]$$

(3)

Example:
- $M = 50$ interferers.
- Annular network.
- $r_{ex} = 0.25$ inner radius.
- $r_{net} = 2$ outer radius.
- $\alpha = 3$.
- $L' = 200$.
- Analytical curves are solid, while $\bullet$ represents simulated values.
Outage Probability

Average Outage Probability for a BPP: Results

- In a binomial point process (BPP), a fixed number of interferers are independently placed according to a uniform distribution.

- The cdf with a BPP network is:

  \[
  F_{Z_M}(z) = \int f_\Omega(\omega)F_Z(z\mid\omega)d\omega \tag{4}
  \]

  where

  \[
  f_\Omega(\omega) = \frac{2\omega^{2/\alpha}}{\alpha \left(r_{net}^2 - r_{ex}^2\right)} \quad \text{for} \quad r_{ex}^\alpha \leq \omega \leq r_{net}^\alpha.
  \]

  Substituting (3) into (4), the cdf of \( Z_M \) is

  \[
  F_{Z_M}(z) = 1 - e^{-\beta\Omega_0^{-1}z} \left[ \frac{\Psi (r_{net}^\alpha) - \Psi (r_{ex}^\alpha)}{r_{net}^2 - r_{ex}^2} \right]^M \tag{5}
  \]

  where

  \[
  \Psi(x) = x^{2/\alpha} \left[ 1 - p + \frac{2p}{\alpha+2} \cdot \frac{x^{2+\alpha/\alpha}}{\beta\Omega_0^{-1}} \cdot 2F_1 \left( \left[ 1, \frac{\alpha+2}{\alpha} \right] ; \frac{2\alpha+2}{\alpha}, -\frac{x}{\beta\Omega_0^{-1}} \right) \right].
  \]
Outage Probability

Average Outage Probability for a BPP: Example

Figure: Outage probability $\epsilon_M$ as a function of $M$ for five values of $L'$ when the location of the nodes is drawn from a BPP. Analytical curves are solid, while • represents simulated values.

Example:

- $r_{ex} = 0.25$.
- $r_{net} = 2$.
- $\alpha = 3$.
- $\beta = 3.7$ dB.
- $\Gamma = 10$ dB.
Average Outage Probability for a PPP: Results

- In a Poisson point process (PPP), the number of interferers $M$ is a Poisson variable.
- The cdf with a PPP network is:
  \[
  F_Z(z) = \sum_{m=0}^{\infty} p_M(m) F_{Z|_m}(z) = \sum_{m=0}^{\infty} \frac{(\lambda A)^m}{m!} e^{-\lambda A} F_{Z|m}(z) \quad (6)
  \]
  where
  - $\lambda$ is the density of the interferers per unit area;
  - $A = \pi (r_{net}^2 - r_{ex}^2)$ is the area of the network;
  - substituting (5) into (6), the average outage probability is:
    \[
    F_Z(z) = 1 - e^{-\beta \Omega_0^{-1} z} e^{-\pi \beta \Omega_0^{-1}} \cdot \{ r_{net}^2 - r_{ex}^2 - \Psi(r_{net}^\alpha) - \Psi(r_{ex}^\alpha) \} \quad (7)
    \]
- When $r_{net} \to \infty$, $r_{ex} = 0$, and $p = L' = 1$:
  \[
  F_Z(z) = 1 - e^{-\beta_0 z} e^{-\frac{2\pi \lambda}{\alpha} \beta_0^{\frac{2}{\alpha}} \Gamma\left(\frac{2}{\alpha}\right) \Gamma\left(1-\frac{2}{\alpha}\right)}
  \]
- The same expression is obtained in literature by Baccelli et al. [4].
Average Outage Probability for a PPP: Examples

Parameters:
- $r_{ex} = 0.25$;
- $r_{net} = 2$;
- $\beta = 3.7 \text{ dB}$;
- $\Gamma = 10 \text{ dB}$.

Parameters:
- $r_{ex} = 0$;
- $\alpha = 3$;
- $\beta = 3.7 \text{ dB}$;
- $\Gamma = 10 \text{ dB}$.
- $L' = 1$;
Outline

1. Frequency-Hopping Ad Hoc Networks
2. Outage Probability
3. Transmission Capacity
4. Modulation Constraints
5. Optimization Results
6. Conclusions
The *transmission capacity* is the spatial spectral efficiency.

If there are $\lambda$ mobiles per unit area, then the number of successful transmissions per unit area is

$$\tau = \lambda (1 - \epsilon)$$

If the outage probability $\epsilon$ is constrained to not exceed $\zeta$, then the transmission capacity is

$$\tau_c(\zeta) = \epsilon^{-1}(\zeta)(1 - \zeta)$$

where $\epsilon^{-1}(\zeta)$ is the maximum mobile density such that $\epsilon \leq \zeta$. 
Transmission Capacity: Results

- For the BPP case, $\epsilon^{-1}(\zeta)$ is found by solving $\epsilon = F_{Z_M}(\Gamma^{-1}) = \zeta$ for \( \lambda = M/A \) and then substituting into (8):

  \[
  \tau_c(\zeta) = \frac{(1 - \zeta) \left[ \log (1 - \zeta) + \beta \Omega_0^{-1} \Gamma^{-1} \right]}{A \log \left\{ (r_{net}^2 - r_{ex}^2)^{-1} \left[ \Psi (r_{net}^\alpha) - \Psi (r_{ex}^\alpha) \right] \right\}}
  \]

- In the PPP case, $\epsilon^{-1}(\zeta)$ is found by solving $\epsilon = F_{Z}(\Gamma^{-1}) = \zeta$ for \( \lambda \) and then substituting into (8):

  \[
  \tau_c(\zeta) = \frac{(1 - \zeta) \left[ \log (1 - \zeta)^{-1} - \beta \Omega_0^{-1} \Gamma^{-1} \right]}{\pi \left\{ r_{net}^2 - r_{ex}^2 - \left[ \Psi (r_{net}^\alpha) - \Psi (r_{ex}^\alpha) \right] \right\}}.
  \]

- Asymptotically for a PPP, with $r_{ex} = 0$ and $p = 1$, as $r_{net} \to \infty$:

  \[
  \tau_c(\zeta) = \frac{(1 - \zeta) \left[ \log (1 - \zeta)^{-1} - \beta \Omega_0^{-1} \Gamma^{-1} \right]}{\pi \beta_0^2 \frac{2\pi}{\alpha} \csc \left( \frac{2\pi}{\alpha} \right)}
  \]

- The same expression is obtained in literature by Weber et al. [14].
Transmission Capacity: Examples

Parameters:
- \( r_{ex} = 0; \)
- \( r_{net} = 2; \)
- \( L' = 1. \)
- \( \beta = -10 \text{ dB}; \)
- \( \alpha = 3; \)

Parameters:
- \( r_{ex} = 0; \)
- \( \Gamma = 10 \text{ dB}. \)
- \( L' = 1; \)
- \( \beta = -10 \text{ dB}; \)
- \( \alpha = 3; \)
Outline

1 Frequency-Hopping Ad Hoc Networks
2 Outage Probability
3 Transmission Capacity
4 Modulation Constraints
5 Optimization Results
6 Conclusions
Until now, we have picked the SINR threshold $\beta$ arbitrarily.

$\beta$ depends on the choice of modulation.

For *ideal* (Gaussian-input) signaling

$$C(\gamma) = \log_2(1 + \gamma)$$

$\beta$ is the value of $\gamma$ for which $C(\gamma) = R$ (the code rate),

$$\beta = 2^R - 1$$

For other modulations, the *modulation-constrained* capacity $C(\gamma)$ must be used, which can be found by measuring the mutual information between channel input and output.

Additionally, the code rate $R$ and the spectral-efficiency of the modulation $\eta$ can be taken into account to give transmission capacity in units of bps/Hz/m$^2$. 

Matthew C. Valenti
Analysis and Optimization of FHAH Network in Rayleigh Fading
May 30, 2012
Noncoherent Binary CPFSK

FH systems often use noncoherent continuous-phase frequency-shift keying, whose capacity & bandwidth depends on the modulation index $h$.

(a) channel capacity versus $E_s/N_0$

(b) bandwidth versus modulation index

The *modulation-constrained* transmission capacity is

\[ \tau' = \lambda (1 - \epsilon) \left( \frac{R \eta(h)}{L'} \right) \]

where

- \( R \) is the rate of the channel code.
- \( h \) is the modulation index.
- \( \eta(h) \) the modulation’s spectral efficiency (bps/Hz).
- \( \epsilon = P[C(\gamma) \leq R] = P[\gamma \leq C^{-1}(R)] \).
- \( L' \) is the effective number of hopping channels.
- \( \lambda \) is the density of interferers.
- \( \tau' \) has units of \( \text{bps}/\text{Hz}/m^2 \).

For a given \( \lambda, \Gamma \), and spatial model, there is a set of \((L', R, h)\) that maximizes \( \tau' \).
Modulation Constraints

Influence of Parameters: BPP

Parameters:
- $r_{ex} = 0.25$;
- $r_{net} = 2$;
- $\Gamma = 10$ dB.
- $M = 50$. 

Matthew C. Valenti  
Analysis and Optimization of FHAH Network in Rayleigh Fading  
May 30, 2012 23 / 31
Influence of Parameters: PPP

Parameters:
- \( r_{ex} = 0.25; \)
- \( r_{net} = 2; \)
- \( \Gamma = 10 \text{ dB}. \)
- \( \alpha = 3. \)
Outline

1. Frequency-Hopping Ad Hoc Networks
2. Outage Probability
3. Transmission Capacity
4. Modulation Constraints
5. Optimization Results
6. Conclusions
Optimization Objectives

- For a given network model \((r_{ex}, r_{net}, \alpha, \lambda)\) and channel SNR \(\Gamma\), the values of \((L', R, h)\) that maximize the modulation-constrained transmission capacity \(\tau'\) are found.
- Optimization is through exhaustive search or a gradient-search strategy.
- Performed for both BPP and PPP spatial models.
### Example of optimization for a BPP

<table>
<thead>
<tr>
<th>$r_{net}$</th>
<th>$r_{ex}$</th>
<th>$\alpha$</th>
<th>$L'$</th>
<th>$R$</th>
<th>$h$</th>
<th>$\tau'_{opt}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.25</td>
<td>3</td>
<td>32</td>
<td>0.62</td>
<td>0.59</td>
<td>0.01590</td>
</tr>
<tr>
<td>3.5</td>
<td>30</td>
<td>0.62</td>
<td>0.59</td>
<td>0.01688</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>28</td>
<td>0.62</td>
<td>0.59</td>
<td>0.01792</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>3</td>
<td>30</td>
<td>0.61</td>
<td>0.59</td>
<td>0.01641</td>
<td></td>
</tr>
<tr>
<td>3.5</td>
<td>29</td>
<td>0.61</td>
<td>0.59</td>
<td>0.01752</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>26</td>
<td>0.59</td>
<td>0.59</td>
<td>0.01871</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.25</td>
<td>3</td>
<td>12</td>
<td>0.54</td>
<td>0.59</td>
<td>0.00983</td>
</tr>
<tr>
<td>3.5</td>
<td>10</td>
<td>0.55</td>
<td>0.59</td>
<td>0.01187</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>0.54</td>
<td>0.59</td>
<td>0.01395</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>3</td>
<td>11</td>
<td>0.52</td>
<td>0.59</td>
<td>0.01024</td>
<td></td>
</tr>
<tr>
<td>3.5</td>
<td>9</td>
<td>0.52</td>
<td>0.59</td>
<td>0.01252</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>0.55</td>
<td>0.59</td>
<td>0.01484</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table:** Results of the Optimization for an annular network area where the interferers are drawn from a BPP. The number of interferers is fixed to $M = 50$. 

Matthew C. Valenti (shortinst)  
Analysis and Optimization of FHAH Network in Rayleigh Fading  
May 30, 2012 27 / 31
### Example of optimization for a PPP

<table>
<thead>
<tr>
<th>$r_{net}$</th>
<th>$r_{ex}$</th>
<th>$\alpha$</th>
<th>$L'$</th>
<th>$R$</th>
<th>$h$</th>
<th>$\tau_{opt}'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.25</td>
<td>3</td>
<td>8</td>
<td>0.62</td>
<td>0.59</td>
<td>0.01597</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3.5</td>
<td>7</td>
<td>0.62</td>
<td>0.59</td>
<td>0.01697</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
<td>7</td>
<td>0.62</td>
<td>0.59</td>
<td>0.01801</td>
</tr>
<tr>
<td>0.5</td>
<td></td>
<td>3</td>
<td>7</td>
<td>0.61</td>
<td>0.59</td>
<td>0.01731</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3.5</td>
<td>7</td>
<td>0.61</td>
<td>0.59</td>
<td>0.01845</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
<td>6</td>
<td>0.59</td>
<td>0.59</td>
<td>0.01973</td>
</tr>
<tr>
<td>4</td>
<td>0.25</td>
<td>3</td>
<td>12</td>
<td>0.52</td>
<td>0.59</td>
<td>0.00977</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3.5</td>
<td>10</td>
<td>0.54</td>
<td>0.59</td>
<td>0.01180</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
<td>8</td>
<td>0.54</td>
<td>0.59</td>
<td>0.01387</td>
</tr>
<tr>
<td>0.5</td>
<td></td>
<td>3</td>
<td>11</td>
<td>0.52</td>
<td>0.59</td>
<td>0.01030</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3.5</td>
<td>9</td>
<td>0.53</td>
<td>0.59</td>
<td>0.01258</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
<td>8</td>
<td>0.55</td>
<td>0.59</td>
<td>0.01491</td>
</tr>
</tbody>
</table>

**Table:** Results of the Optimization for an annular network area where the interferers are drawn from a PPP. The intensity $\lambda$ per unit area is fixed to $\lambda = 1$. 

---

Matthew C. Valenti (shortinst)
Analysis and Optimization of FHAH Network in Rayleigh Fading May 30, 2012 28 / 31
1. Frequency-Hopping Ad Hoc Networks
2. Outage Probability
3. Transmission Capacity
4. Modulation Constraints
5. Optimization Results
6. Conclusions
The performance of frequency-hopping ad hoc networks is a function of many parameters.
- Number of hopping channels $L$.
- Code rate $R$.
- Modulation index $h$ (if CPFSK modulation).

These parameters can be jointly optimized.
- Transmission capacity is the objective function of choice.
- The modulation-constrained TC quantifies the tradeoffs involved.

The approach is general enough to handle a wide variety of conditions
- Can be extended to accommodate Nakagami fading and shadowing.
- Any spatial model, including repulsion models.
- Adjacent-channel interference due to spectral splatter.
- Adaptive code rates ($R$ not fixed for all users).

Additional constraints can be imposed on the optimization.
- Per-node outage constraint.
- Fixed or minimum data rates per user.
Thank You