Problem: Computing the Spatially Averaged Outage Probability

\( M \) interferers are randomly distributed within an arbitrary region \( \mathcal{A} \), where \( M \) can be fixed (e.g., BPP) or random (e.g., PPP).

- The SINR at the reference receiver is:
  \[
  \gamma = \frac{g_r}{\text{SNR}^{-1} + \sum_{i=1}^{M} I_i g_i r_i^{-\alpha}}
  \]
  where:
  \( \{r_i\} \) are the distances to the \( M \) interferers.
  \( \{g_i\} \) are the fading gains (e.g., i.i.d. exponential).
  \( \{I_i\} \) are Bernoulli variables indicating interference (\( P[I_i = 1] = p \)).
  \( \alpha \geq 2 \) is the path-loss exponent.

- The goal is to find the spatially averaged outage probability:
  \[
  \epsilon = \mathbb{E}_M [P[\gamma \leq \beta]]
  \]
  where \( \beta \) is the SINR threshold.

- We would like to do this directly; i.e., without resorting to transforms such as a Laplace transform.

The Direct Approach to Spatial Averaging [1]

- Fix the \( r \) and find conditional outage probability by averaging over just the fading
  \[
  \epsilon (r) = P[\gamma \leq \beta | r]
  \]
  \[
  \epsilon [M] = \mathbb{E}_r [\epsilon (r) | M] = \int \epsilon (r) f_r (r) d r
  \]
  where the integral may be computed for an arbitrarily distributed \( r \) as
  \[
  \int f_r (r) \frac{\gamma_0}{\beta + \gamma_0} d r = \mathbb{E}_r \left[ \frac{-\ln \gamma_0}{\beta + \gamma_0} \right].
  \]

- If BPP, can stop, since \( M \) is fixed.

- If \( M \) is random, then take expectation with respect to the \( M \).

As a final step, can take the limit as the area extends to the infinite plane.

Specific Cases

- In Rayleigh fading [2, 3],
  \[
  \epsilon (r) = 1 - e^{-\beta/\text{SNR}} \prod_{i=1}^{M} \left( 1 - p + \frac{p r_i^{\alpha}}{\beta + r_i^{\alpha}} \right).
  \]
  When the \( M \) interferers are i.i.d. over \( \mathcal{A} \) (i.e., a binomial point process):
  \[
  \epsilon [M] = 1 - e^{-\beta/\text{SNR}} \prod_{i=1}^{M} \left( 1 - p + \frac{p r_i^{\alpha}}{\beta + r_i^{\alpha}} \right)
  \]
  which the integral may be computed for an arbitrarily distributed \( r \) as
  \[
  \int f_r (r) \frac{\gamma_0}{\beta + \gamma_0} d r = \mathbb{E}_r \left[ \frac{-\ln \gamma_0}{\beta + \gamma_0} \right].
  \]
  For irregular shapes this can be found through a Monte Carlo simulation.

- When \( \mathcal{A} \) is a regular polygon, the solution is given by [1, 4].

- When \( \mathcal{A} \) is a disk of radius \( r_0 \) and \( M \) is fixed, then:
  \[
  \epsilon [M] = 1 - e^{-\beta/\text{SNR}} \left( 1 + \frac{p r_0^{\alpha}}{\beta + r_0^{\alpha}} \right)^M
  \]
  where:
  \[
  \Phi (x) = \frac{\alpha}{\beta + \alpha} F_1 \left( \frac{2}{\beta + \alpha}, \frac{1}{2}; \frac{2}{\beta + \alpha} \right)\frac{x^{\beta + \alpha}}{\alpha^{\beta + \alpha}}.
  \]
  When \( M \) is Poisson (a Poisson point process defined over \( \mathcal{A} \)), then:
  \[
  \epsilon = 1 - e^{-\beta/\text{SNR} + \pi \lambda \Phi (r_0)}
  \]
  where \( \lambda \) is the intensity of the PPP.

Outage for Finite Regions

The coverage probability \( 1 - \epsilon \) for three shapes and the following parameters: \( \alpha = 3.2, \beta = 0 \) dB, \( \text{SNR} = 10 \) dB, \( r_0 = 2 \), and \( |\mathcal{A}| = 4\pi \).

Revisiting Baccelli

To obtain the outage probability when the interferers are drawn from a PPP defined over the entire plane, take the limit of (1) as \( r_0 \to \infty \),

\[
\epsilon = 1 - \exp \left\{ -\frac{\beta}{\text{SNR}} + \pi \lambda \lim_{y \to \infty} \Phi (y) \right\}.
\]

By performing the change of variables \( y = \frac{\beta}{\beta} \)

\[
\lim_{y \to \infty} \Phi (y) = \lim_{y \to \infty} \left\{ \frac{\alpha y^{\gamma_0/\beta}}{\gamma_0} \right\} = \frac{2 \alpha y^{\gamma_0/\beta}}{\gamma_0} \}
\]

where

\[
\phi (y) = \frac{1}{y} \int_0^y \left( 1 + \frac{t}{y} \right)^{-1} dt - \frac{\alpha}{2}.
\]

It is shown in [1] that

\[
\lim_{y \to \infty} y^{\gamma_0/\beta} \phi (y) = - \sum_{i=0}^{\infty} \frac{(-1)^i}{i + 1} = - \pi \csc \left( \frac{2 \pi}{\alpha} \right).
\]

Substituting (4) into (3) and (2) yields

\[
\epsilon = 1 - \exp \left\{ -\frac{\beta}{\text{SNR}} + \pi \lambda \frac{2 \alpha y^{\gamma_0/\beta}}{\gamma_0} \phi (y) \right\}
\]

which for \( \text{SNR} \to \infty \) corresponds to equation (3.4) in [5].

Benefits and Conclusions

- The spatially outage probability can be found without needing to resort to Laplace transforms or the probability generating functional (pgfl).
- The approach is particularly effective for finding outage probability when the network is a BPP or PPP defined over a finite region.
- Through a simple 1-dimensional Monte Carlo simulation, any arbitrary shape can be handled.
- In the limit of a PPP defined over an infinite plane, the result of Baccelli is retrieved.

References


