Distributed Turbo Codes: Towards the Capacity of the Relay Channel

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Abstract— **A novel coding technique is proposed for the relay channel. The source broadcasts a recursive convolutional code to both relay and destination. After detecting the data broadcasted by the source, the relay interleaves and re-encodes the message prior to forwarding it to the destination. Because the destination receives both codes in parallel, a distributed turbo code is embedded in the relay channel. Simulation results show that the proposed code performs close to the information-theoretic bound on outage event probability of decode-and-forward relaying.**

I. INTRODUCTION

In point-to-point links, spatial diversity is typically implemented with antenna arrays at the transmitter and/or receiver, possibly in conjunction with space time coding. For certain types of ad hoc networks, such as wireless sensor networks, antenna arrays are too cumbersome to be employed at each device. However, the low device costs associated with these networks allows the coverage area to be blanketed with a dense deployment of devices. The performance of these densely deployed wireless networks can be improved by leveraging the intrinsic spatial diversity due to the presence of multiple devices [1].

Distributed diversity can be implemented by the use of *relaying.* A *relay channel* is a three terminal network consisting of a source, a relay, and a destination [2]. The source broadcasts a message to both relay and destination, while the relay forwards the message to the destination. Note that relaying is different from standard multihop routing in that communications is no longer point-to-point. Instead, each transmitter *broadcasts* its message, which is received by any node able to act as a relay. Each receiver then collects the transmissions of not only the source, but also all the forwarding relays, thereby achieving a diversity effect. The concept could be considered similar to frequency-hopping, whereby transmissions occur at different carrier frequencies. Now, however, the transmissions occur from different spatial locations, and therefore could be described by the term *spatial-hopping.*

A twist on this idea called *cooperative diversity* involves networks with two sources (users), each of which may also serve as a relay for the other user [3], [4], [5], [6]. Users pair up and act cooperatively to convey information to a common destination, for instance a cellular base station. Time is slotted, and during the first slot the two users each broadcast their message over orthogonal channels (which could be implemented through time-, frequency-, or code-division multiple access). After the first slot, each user attempts to decode the other's message. If a particular user can decode the other's message, then it forwards the other user's information to the destination during the second slot. The forwarded message could either be identical to the initial transmission (repetition coding), or it could be a different part of a rate compatible code (incremental redundancy). If the user is unable to decode, then it can either amplify-and-forward its noisy version of the other user's message or it could resend its own information, again either using repetition coding or incremental redundancy.

In addition to the recent research on the theoretic performance limits of various relaying protocols, there have been some developments on the topic of coding for the relay channel. In [7], Laneman and Woernell proposed *distributed* space-time code for the relay channel (possibly with multiple relays) and showed its benefits from an information theoretic standpoint. A method for achieving cooperative diversity using rate compatible convolutional (RCPC) codes (i.e. the incremental redundancy approach) was proposed in [6], [8] and termed *user cooperative coding.* In [9], we first proposed the distributed turbo coding technique discussed in more detail in this paper.

In this paper, we further discuss the concept of distributed turbo coding. In addition, we derive an appropriate information theoretic performance bound, and compare the performace of our coding technique and a RCPC-based technique against it. We extend our results to exploit the presence of multiple relays, in which case the turbo code is a distributed *multiple* turbo code.

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II. PERFORMANCE LIMITS

In [2], bounds on the capacity of the relay channel in AWGN were found along with exact expressions for certain cases. However, the only constraint imposed was on the average transmit power at the source and relay. In order to achieve the promised capacity, the relay would need to simultaneously receive and transmit over the same channel. Furthermore, the source and relay would need to transmit coherently, thereby achieving a beamforming effect. However, in practical systems, it is more realistic to assume that (a) the relay receives and transmits in different time slots in a causal fashion; (b) the relay and the source transmit over orthogonal channels (since it is not generally practical to adjust the transmit phases at the source and relay so that they are identical at the destination). For instance, in many modern systems, such as 3-G cellular in time-division duplexing (TDD) mode, the relay might not be able to simultaneously receive and transmit. In general, soure/relay orthogonality could be achieved through time-, frequency-, or code-division multiple access or by using orthogonal space-time codes. The capacity under the TDD constraint (a) was analyzed in [10]. As in [2], the source and relay were allowed to transmit coherently, thereby achieving a beamforming effect. However because the source and relay usually have separate oscillators, it is difficult to make their phases add coherently at the destination. Thus we impose the additional constraint (b). Furthermore, we require that the relay decode its received message and re-encode prior to forwarding (*decode-and-forward*). We assume that the relay can perform perfect error detection and will not forward if it cannot decode. While the orthogonal decode-and-forward relaying restriction may reduce the achievable performance, it is a better model of practical systems and allows for a more straightforward capacity analysis.

We assume a quasi-static Rayleigh fading channel. Each rate r codeword is divided into one or more *blocks*. Each block is sent as a *burst* during a particular time *slot*. The channel between two terminals is AWGN for the duration of a particular block, but the SNR changes from block-to-block. In particular, the sequence $\{\gamma\}$ of block SNRs is i.i.d. exponential, with average received SNR Γ. Typically, we assume that the codeword is divided into two blocks which need not be of equal size. During the first time slot, the source transmits its first block. If the relay can decode the entire codeword on the basis of this first block, then it will recalculate the codeword and send just the second block to the destination. We denote by α the fraction of time that the source transmits and by $\bar{\alpha}$ the fraction of time that the relay transmits. Note that this is equivalent to transmission over a conventional (point-to-point) block-fading channel [11], only now the blocks could be transmitted from different radios (i.e. *space-hopping*). If multiple relays are used then the codeword could be divided into more than just two blocks.

First consider just a single block transmitted over a point-to-point link. During this block, the channel is AWGN with instantaneous SNR γ and instantaneous capacity $C(\gamma) = \frac{1}{2}\log_2(1+\gamma)$. If a rate r code is used, then the channel will be in an *outage* whenever $C(\gamma) < r$, where $\{C(\gamma) < r\}$ is called the *outage event*. The *outage event probability* (OEP) is found by integrating the pdf of γ over the outage event region,

$$\begin{aligned} P_o &= \int_0^{C^{-1}(r)} p(\gamma) d\gamma \\ &= \int_0^{2^{2r}-1} \frac{1}{\Gamma} \exp\left\{\frac{-\gamma}{\Gamma}\right\} d\gamma \\ &= 1 - \exp\left\{\frac{-(2^{2r}-1)}{\Gamma}\right\} \end{aligned} \quad (1)$$

Now consider the relay channel. Computation of the OEP is complicated by the fact that now there are three SNRs, $\gamma_{s,r}, \gamma_{s,d}$, and $\gamma_{r,d}$, corresponding to the source-relay, source-destination, and relay-destination links, respectively. During the first slot, the source transmits a rate r/α code, while during the second slot, the relay transmits a rate $r/\bar{\alpha}$ code. The relay is in an outage if $C(\gamma_{s,r}) < r/\alpha$. When the relay is in an outage, an end-to-end outage occurs if the source-destination link is also in an outage, $C(\gamma_{s,d}) < r/\alpha$. On the other hand, if the relay is not in an outage, then the destination will receive a transmission from both source and relay. The source and relay are then transmitting over orthogonal parallel Gaussian channels. Since capacity adds for parallel channels [12], an outage will occur if $\alpha C(\gamma_{s,d}) + \bar{\alpha} C(\gamma_{r,d}) < r$. Thus, the outage event for the relay channel with orthogonal decode-and-forward relaying is

$$E_o = \left\{ \left[\left(C(\gamma_{s,r}) < \frac{r}{\alpha} \right) \cap \left(C(\gamma_{s,d}) < \frac{r}{\alpha} \right) \right] \right.$$

$$\left. \cup \left[\left(C(\gamma_{s,r}) > \frac{r}{\alpha} \right) \cap \left(\alpha C(\gamma_{s,d}) + \bar{\alpha} C(\gamma_{r,d}) < r \right) \right] \right\}. \quad (2)$$

The end-to-end OEP is found by integrating $p(\gamma_{s,d}, \gamma_{s,r}, \gamma_{r,d})$ over the area defined by E_o,

$$
\begin{aligned}
P_o &= \left(1 - \exp \frac{1 - 2^{2r/\alpha}}{\Gamma_{s,r}}\right)\left(1 - \exp \frac{1 - 2^{2r/\alpha}}{\Gamma_{s,d}}\right) \\
&\quad + \int\int_{\mathcal{A}} \frac{\exp \frac{1-2^{2r/\alpha}}{\Gamma_{s,r}}}{\Gamma_{s,d}\Gamma_{r,d}} \exp\left\{-\frac{\gamma_{s,d}}{\Gamma_{s,d}} - \frac{\gamma_{r,d}}{\Gamma_{r,d}}\right\} d\gamma_{s,d} d\gamma_{r,d} \\
&= \left(1 - \exp\left\{\frac{1 - 2^{2r/\alpha}}{\Gamma_{s,d}}\right\}\right) \\
&\quad - \exp\left\{\frac{1 - 2^{2r/\alpha}}{\Gamma_{s,r}}\right\} \int_0^{2^{2r/\alpha}-1} \Phi\left(r, \alpha, \Gamma_{s,d}, \Gamma_{r,d}\right) d\gamma_{s,d}
\end{aligned}
\tag{3}
$$

where $\mathcal{A} = \{(1+\gamma_{s,d})^{\alpha}(1+\gamma_{r,d})^{\bar{\alpha}} < 2^{2r}\}$ and

$$
\Phi\left(r, \alpha, \Gamma_{s,d}, \Gamma_{r,d}\right) = \frac{1}{\Gamma_{s,d}} \exp\left\{-\frac{\gamma_{s,d}}{\Gamma_{s,d}} - \frac{1}{\Gamma_{r,d}}\left(\frac{2^{2r/\bar{\alpha}}}{(1+\gamma_{sd})^{\alpha/\bar{\alpha}}} - 1\right)\right\}
$$

III. Distributed Turbo Coding

We now turn our attention to a practical method that performs close to the information theoretic bounds. First consider the classic single-relay channel. The source and relay each employ a very simple code, in this case a two-state rate 1/2 recursive systematic convolutional (RSC) code with octal feedback and feedforward generators $(3, 2)_8$, respectively. After encoding, the signal is BPSK modulated. Note that the parity output of this code is generated by a simple differential encoder (accumulator), so an alternative interpretation is that each terminal transmits BPSK and DPSK in parallel (see Fig. 1). A conventional decode-and-forward relay with repetition coding will detect the RSC encoded signal and re-encode it with an identical RSC encoder. The destination will receive two versions of the same code word, one directly from the source and the other from the relay. The two signals may be MRC combined and the information bits detected with a Viterbi decoder.

The new twist in our proposed scheme is to add an *interleaver* to the relay, as shown in Fig. 1. If the relay interleaves its estimate of the source's data prior to RSC encoding, then the source and relay have cooperatively constructed a *distributed* turbo code. Recall that with a turbo code, or parallel concatenated convolutional code (PCCC), the data is recursively encoded twice, first in its natural order and again after being interleaved [13]. Thus, the uninterleaved encoding is present in the source-destination path, while the interleaved encoding is present in the relay-destination path. The destination can detect the code iteratively by using a standard turbo decoder [13]. Although the turbo decoder adds some complexity at the destination, the complexity is still reasonable since the constituent encoders only have two states. While this construction maintains the diversity benefit of relaying, the coding gain is far superior than that of a single RSC observed over two independent channels. This extra coding gain is due to the *interleaving* gain of the turbo code construction and the *turbo processing* gain of the iterative decoder.

The concept of distributed turbo coding is a rather broad. For instance, if the relay re-encodes the detected data with a (3, 2)8 RSC encoder prior to interleaving and DPSK encoding, then the relay channel contains a distributed *serial* concatenated convolutional code (SCCC). Moreover the proposed coding approach could be naturally applied in the channel with multiple relays. In particular, if each relay were to interleave the received data prior to encoding and according to a unique interleaving pattern, then the result would be a distributed *multiple* turbo code [14]. Thus, the performance can be further improved by collecting additional diversity benefit and more interleaving gain.

IV. Simulation Results

A simulation campaign was carried out to investigate the performance characteristics of distributed turbo coding. In all simulations, data was grouped into frames of length 512 bits. Initially the simple rate 1/2 $(3, 2)_8$ RSC code was used to construct a distributed turbo code. Later, more complex RSC codes were used in the code construction to further improve performance.

A. Transmit SNR Contour

Consider a system with a relay located between a source and destination separated by 10 m. Assuming a transmit frequency of $f_c = 2.4$ GHz, a path loss coefficient $n = 3$, and a free-space reference distance $d_o = 1$ m, the average received power at receiver $j \in \{r, d\}$ is $P_j^{(r)} = K_o(d_{ij})^{-n} P_i^{(t)}$, where $K_o = (c/4\pi d_o f_c)^2 \approx 10^{-4}$, d_{ij} is the transmitter-receiver separation, and $P_i^{(t)}$ is the transmitted power of node $i \in \{s, r\}$. The receive SNR at node j is $\Gamma_s^{(r)} = P_s^{(r)}/N$ where N is the noise power (assumed constant at all receivers). For convenience, we define the *transmit SNR* to be $\Gamma_s^{(t)} = P_s^{(t)}/N$.

In the simulations, the transmit SNR of the source ($\Gamma_s^{(t)}$) and relay ($\Gamma_r^{(t)}$) were varied independently, and it was noted which $(\Gamma_s^{(t)}, \Gamma_r^{(t)})$ pairs achieved a target source-destination frame error rate (FER) of 10^{-2}. We consider two network topologies: (i) the relay is located halfway between the source and destination, and (ii) the relay is 1 m away from the source and 9 m away from the destination. The corresponding simulation results are shown for tolopogies (i) and (ii) in Fig. 2 and Fig.

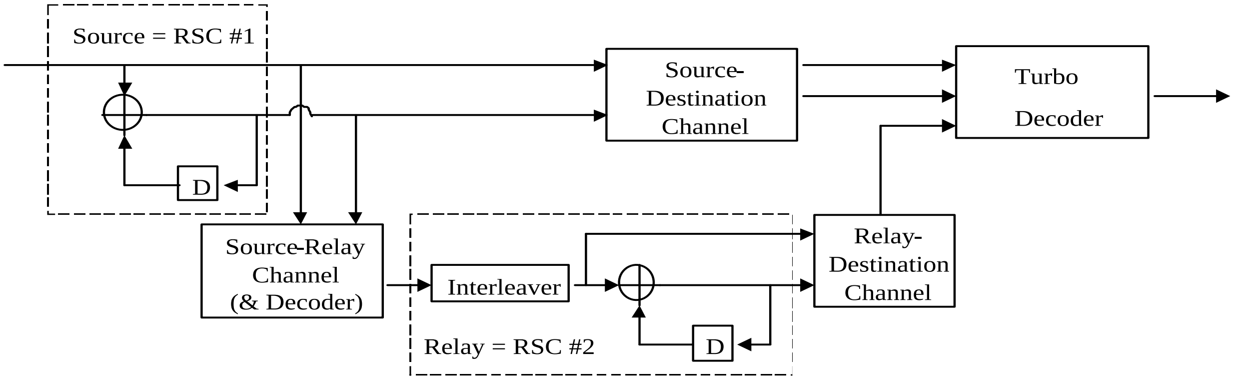

Fig. 1. When an interleaver separates source from relay, the relay channel contains a turbo code.

3, respectively. In particular, we compare the performance of several coding strategies against that of the theoretical bound for decode-and-forward with $r = 1/4$ and $\alpha = 1/2$. Note that results for the same five codes shown in Fig. 2 were previously given for topology (i) in [9], but here we have added the theoretical bound to provide a frame of reference. The curves labelled *Distributed rate 1/4 PCCC* and *Simple PCCC code* correspond to the code shown in Fig. 1. For the curve labelled *RSC* relay, there is no interleaver at the relay. For the curve labelled *Distributed rate 1/3 PCCC* the relay only sends its parity output. For the curves labelled *Distributed rate 1/4 SCCC* and *Simple SCCC code* the relay re-encodes the detected message bits with the rate 1/2 RSC prior to interleaving. In addition, results for two stronger codes are shown in Fig. 3: a rate *1*/4 PCCC with generator polynomials $(13, 15)_8$ labelled *Stronger PCCC code*, and a rate *1*/4 SCCC with an inner code polynomial $(3, 2)_8$ and an outer code polynomial $(23, 35)_8$ labelled *Stronger SCCC code*. Fig.3 indicate that with stronger codes, distributed turbo coding could achieve an extra 2 dB gain in energy efficiency, and therefore it could come within 2.5 dB of the theoretic bound.

As a comparison, we study the performance of the rate 1/4 RCPC code with generator $(23, 35, 27, 33)_8$ used for cooperative coding in [8]. The contour curve in Fig. 3 indicates that the RCPC code outperforms the simplest distributed PCCC by 1.25 dB. However, it is still 5.5 dB away from the theoretic limit and the stronger PCCC code outperforms it by about 3 dB.

Comparing the performance of the two topologies, we observe that when the relay is closer to the source, the minimum Γ_s required by the theoretic bound has a wider dynamic range. Also, we note that the curves for distributed SCCC and distributed PCCC are more consistently close to the theoretic bound over a wider range of SNRs than they were when the relay was centrally located.

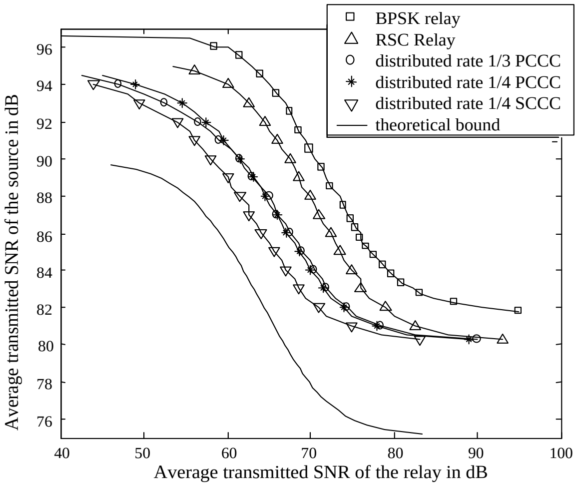

Fig. 2. Minimum transmit SNR at source and relay required to achieve an end-to-end FER of 10^{-2} when the relay is halfway between the source and destination.

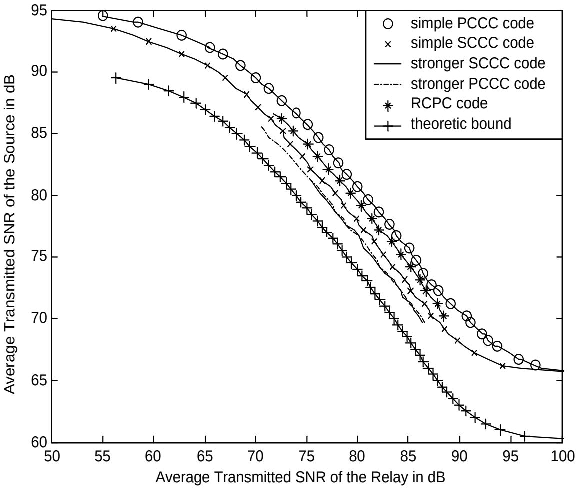

Fig. 3. Minimum transmit SNR at source and relay required to achieve an end-to-end FER of 10^{-2} when the relay is 1 m away from the source and 9 m away from the destination.

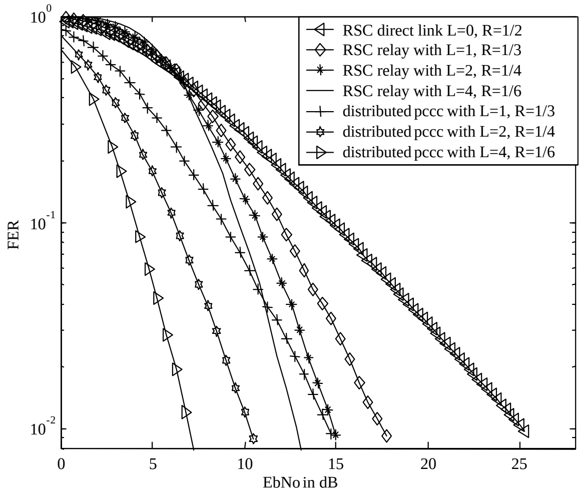

Fig. 4. FER for distributed multiple turbo codes over the multiple relay channel under the assumption of perfect source-relay links.

B. Multiple Relay Channel

When the concept of distributed turbo coding is extended to the multiple relay channel, the result is a distributed *multiple* turbo code. Thus the code constructions and decoding stategies discussed in [14] can be readily applied. In our simulations, we assume that the broadcast channel from source to the multiple relays are always reliable (which is likely when the relays are clustered close to the source), and that through perfect power control, the destination's average received SNR from the source and multiple relays are identical.

A total of seven scenarios were simulated, one for a direct RSC encoded transmission (no relay), and then a pair of simulations for each of L = 1, 2, and 4 relays. When multiple relays are present, two strategies were simulated, an RSC repetition code (each relay uses the same RSC encoder and no interleaver; MRC combining and Viterbi decoding is performed at the destination) and a parallel multiple turbo code (each relay interleaves the decoded data with a unique interleaving pattern before differential encoding; a turbo decoder with L+1 soft-in/soft-out modules is used at the destination). In each case, the RSC encoders use generator $(3,2)_8$ and the relays only transmit the parity output. Because the source broadcasts a rate 1/2 RSC and each relay transmits a rate 1 code, the overall code rate is $R = \frac{1}{L+2}$. By varying L, we can use the direct transmission results to benchmark the pure diversity gain due to the multiple relay paths, so that the additional interleaving gain of using a multiple turbo code can be easily isolated.

With multiple RSC relaying, a total of 4.7 dB extra diversity gain is be achieved when the number of relays L increases from 1 to 4. With distributed multiple turbo codes, the additional interleaving gain increases from 3 dB for single relay ($L = 1$) to nearly 6 dB for multiple relays ($L = 4$). Compared with the RSC encoded direct link transmission, distributed multiple turbo coding ($L = 4$) could achieve a total of 18 dB 'cooperative coding' gain. Both diversity gain and interleaving gain tend to yield diminishing marginal benefit with each additional relay.

V. Conclusions

A simple, but efficient, coding technique has been developed for the quasi-static relay channel. An appropriate information theoretic bound on capacity was derived, and it was shown that the proposed code comes within 2.5 dB of this bound. A performance comparison between distributed turbo codes and the RCPC code indicates that, with extra interleaving gain, distributed turbo coding is 3 dB more efficient than RCPC code at an end-to-end $FER = 10^{-2}$. Moreover, the extension of distributed turbo coding to the multiple relay channel has been investigated and a significant performance advantage over the conventional relaying method demonstrated.

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