The Signals & Systems Workbook

A companion to EE 329

Version 1.2

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Preface

This workbook was developed at West Virginia University during the Spring 2002 semester as an accompaniment to EE 329, Signals & Systems II. It was written in response to the difficulty we had finding an appropriate text. The EE 329 syllabus covers a wide range of topics, including the Fourier Transform, probability, and elementary modulation and control applications. As no single book covered all of these topics, we choose to carryover the book used by Signals & Systems I (EE 327) and then use supplemental material, including this workbook. In the Spring 2003, this book was used by Dr. Jerabek and I thank him for his feedback.

One thing you will notice is that this is no ordinary textbook. There are lots of "holes" in the text. These holes should be filled out by the student, either in class (led by the instructor) or while studying outside of class. Enough information is given that an astute student with reasonable mathematical ability should be able to fill in the empty spaces in the workbook on his or her own. However, perhaps the most effective way to use this text is for the student to bring it to lecture and fill it out there. The role of the instructor is then to help guide the student through the book. The student response to this workbook approach was very positive, as it allowed more time to be devoted to important derivations and examples.

One other thing: This is a work in progress. There are bound to be several typos or other mistakes. Catching a mistake is a good sign that you are paying attention! If you find a mistake in this book, please let me know by emailing me at mvalenti@wvu.edu.

Contents

Chapter 1

Signals

1.1 Definition of a Signal

A **signal** is a function of one or more independent variables. A **one-dimensional** signal has a single independent variable, while a **two-dimensional** signal has a second independent variable. *Can you give an example of a one-dimensional signal? A two-dimensional signal?*

In this book, we will usually only consider one-dimensional signals and the independent variable will usually be either time (t) or frequency (f). We will often use transformations to go between the time-domain and frequency-domain.

If the independent variable is time, then the signal can be either **continuous time** or **discrete time**, depending on whether it is defined for all possible time instances or only at specific times. *Give an example of a continuous-time and a discrete-time signal.*

1.2 Unit Step Function

The unit step function $u(t)$ is defined as follows:

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \tag{1.1}$$

What about for $t = 0$*?*

1.3 Delay

What happens if we change the argument of the step function?

Sketch the signal $u(t-2)$

For a general function $x(t)$ what is $x(t-t_o)$?

1.4 Rectangular Pulse

The rectangular pulse $\Pi(t)$ is defined as:

$$\Pi(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & |t| > \frac{1}{2} \end{cases} \tag{1.2}$$

Express the rectangular function in terms of the unit step function.

Now sketch the delayed signal $\Pi\left(t-\frac{1}{2}\right)$.

1.5 Time Scaling

Now let's change the argument of the step function, but in a different way.
Sketch the signal $\Pi\left(\frac{t}{2}\right)$.

Sketch the signal $\Pi\left(\frac{t}{T}\right)$.

In general how is $x(t)$ *related to* $x\left(\frac{t}{T}\right)$*?*

1.6 Putting It Together: Delay and Scaling

What if we *delay* and *scale* the time axis?

Sketch the specific signal $\Pi\left(\frac{t-2}{2}\right)$

Sketch the generic signal $\Pi\left(\frac{t-t_o}{T}\right)$

1.7 Adding Signals

Think about how signals add? *Sketch* $x(t) = \Pi(t) + \Pi\left(\frac{t}{2}\right)$.

1.8 Multiplying Signals

Now think about how signals multiply. *Sketch* $x(t) = \Pi(t)\Pi\left(\frac{t}{2}\right)$.

Let $y(t) = \Pi\left(\frac{t}{A}\right)$ and $z(t) = \Pi\left(\frac{t}{B}\right)$. Sketch the signal $x(t) = y(t)z(t)$.

1.9 The Triangular Function

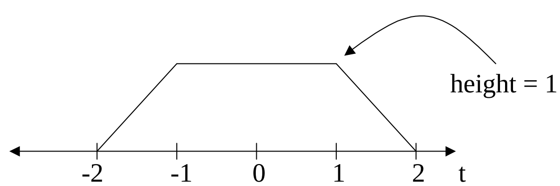
Another interesting function is the triangular function:

$$\Lambda(t) \quad = \quad \begin{cases} 1-|t| & |t|<1 \\ 0 & |t|>1 \end{cases} \tag{1.3}$$

Sketch this function.

1.10 Subtracting signals

Express the following signal in terms of triangular functions:



1.11 Time reversal:

Sketch $u(-t)$

Sketch $u(t_o - t)$

Generalize: $x(t_o - t)$

1.12 Integration

Integrating a signal is equivalent to finding the *area* under the curve.

Compute

$$X = \int_{-\infty}^{\infty} x(t)dt$$

for

$\Pi(t)$

$\Pi\left(\frac{t}{T}\right)$

$\Lambda\left(\frac{t}{T}\right)$

1.13 Convolution:

Convolution is defined as:

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\lambda)x_2(t-\lambda)d\lambda \qquad (1.4)$$

Convolution is used to express the input-output relationship of linear time invariant (LTI) systems. In particular, if the input to an LTI system is $x(t)$ and the **impulse response** of the system is $h(t)$, then the output of the system is $y(t) = x(t) * h(t)$.

Properties of convolution:

Commutative:

Distributive:

Associative:

Linearity:

Example: Find and sketch the signal $y(t) = \Pi(t) * \Pi(t)$.

1.14 The Delta Function:

The continuous time *unit impulse* or *dirac delta* function $\delta(t)$ is the time derivative of the unit step function:

$$\delta(t) = \frac{du(t)}{dt} \tag{1.5}$$

Alternatively, the delta function can be defined as the function that satisfies *both* of the following two conditions:

1. $\delta(t) = 0$ for $t \neq 0$.
2. $\int_{-\infty}^{\infty} \delta(t) dt = 1$.

These conditions are satisfied by $\frac{1}{T}\Pi\left(\frac{t}{T}\right)$ as $T \to 0$.

1.14.1 Properties of the Delta Function

The delta function has the following properties:

Even function: $\delta(t) = \delta(-t)$.

Time scaling: $\delta(t/T) = |T|\delta(t)$.

Integral: For any $\epsilon > 0$, $\int_{-\epsilon}^{\epsilon} \delta(t) dt = 1$.

Multiplication with another function: $g(t)\delta(t) = g(0)\delta(t)$.

Sifting property: $\int_{-\infty}^{\infty} g(t)\delta(t) dt = g(0) \int_{-\infty}^{\infty} \delta(t) dt = g(0)$.

Now think about what the properties of $\delta(t - t_o)$ would be:

1. For any $\epsilon > 0$, $\int_{t_o-\epsilon}^{t_o+\epsilon} \delta(t - t_o) dt =$
2. $g(t)\delta(t - t_o) =$
3. $\int_{-\infty}^{\infty} g(t)\delta(t - t_o) dt =$

1.14.2 Convolution with Delta Functions

Find $y(t) = x(t) * \delta(t)$.

Now find $y(t) = x(t) * \delta(t - t_o)$

Apply this result to find $y(t) = \delta(t - 1) * \delta(t - 2)$

1.15 Exercises

1. Sketch each of the following signals:

$$
\begin{aligned}
x_1(t) &= u\left(t+\frac{1}{2}\right) - u\left(t-\frac{1}{2}\right) \\
x_2(t) &= \Pi\left(\frac{t}{3}\right) + \Pi\left(\frac{t}{6}\right) \\
x_3(t) &= \sum_{k=-\infty}^{\infty} \delta(t-k) \\
x_4(t) &= x_2(t)x_3(t) \\
x_5(t) &= x_3(t)\text{sinc}(t) \\
x_6(t) &= 2\Lambda\left(\frac{t-1}{2}\right) + \Lambda(t-1) \\
x_7(t) &= \Pi\left(\frac{t}{3}\right)\Pi\left(\frac{t}{6}\right) \\
x_8(t) &= \Pi\left(\frac{t-2}{2}\right) + \Lambda\left(t-2\right) \\
x_9(t) &= x_8(t)\sum_{k=-\infty}^{\infty} \delta\left(t-\frac{k}{2}\right)
\end{aligned}
$$

2. Represent $\Pi(t/T)$ as:

(a) The difference of two time-shifted unit step functions.

(b) The product of two unit step functions, both of which are time-shifted and one of which is reversed in time.

3. Find the numerical value of X for each of the following:

(a) $X = \int_{-\infty}^{\infty} \Pi\left(\frac{t-2}{2}\right) dt.$

(b) $X = \int_{-\infty}^{\infty} \left[2\Lambda\left(\frac{t}{2}\right) - \Lambda\left(t\right)\right] dt.$

(c) $X = \int_{-\infty}^{\infty} \delta\left(\frac{t}{2}\right) dt.$

(d) $X = \int_{-\infty}^{0} \delta\left(t-2\right) dt.$ (Pay close attention to the limits)

(e) $X = \int_{-\infty}^{\infty} \Lambda\left(\frac{t}{4}\right) \delta\left(t-1\right) dt.$

4. Let the input $x(t)$ and impulse response $h(t)$ of a linear time invariant (LTI) system be:

$$x(t) = \Lambda\left(\frac{t}{2}\right)$$
$$h(t) = 2\delta(t-2)$$

a. Find and sketch $y(t) = x(t) * h(t)$, where $*$ denotes convolution.

b. Calculate the value of:

$$Y = \int_{-\infty}^{\infty} y(t)dt$$

5. Perform the following convolutions (in the time domain) and sketch your result:

(a) $y(t) = \Pi(t) * (\delta(t-4) - \delta(t+4))$.

(b) $y(t) = \Pi(t-1) * \Pi(t-2)$.

(c) $y(t) = \Pi\left(\frac{t}{2}\right) * \Pi\left(\frac{t}{4}\right)$.

6. Let the input $x(t)$ and impulse response $h(t)$ of a linear time invariant (LTI) system be:

$$x(t) = \Pi\left(\frac{t-1}{2}\right)$$
$$h(t) = \delta(t) - \delta(t-1)$$

Find and sketch $y(t) = x(t) * h(t)$, where $*$ denotes convolution.

Chapter 2

The Fourier Series

2.1 Periodic and Aperiodic Signals

A signal x(t) is *periodic* if there exists a positive constant T such that

$$x(t) = x(t+T) \tag{2.1}$$

for all values of t. The smallest value of T for which this is true is called the *fundamental period* and is denoted T_o. The corresponding *fundamental frequency* is $f_o = \frac{1}{T_o}$. If T_o is in seconds, then f_o is in Hertz (Hz). The *fundamental angular frequency* is $\omega_o = 2\pi f_o$ and is measured in rad/sec.

If no value of T can be found that satisfies (2.1) for all t, then the signal is *aperiodic*.

2.2 Energy and Power

2.2.1 Instantaneous Power

Consider an electrical signal over a resistor with resistance R ohms. Let $v(t)$ be the voltage across the resistor and $i(t)$ be the current through the resistor. Then from Ohm's law, the *instantaneous power* is:

$$\begin{aligned} p(t) &= v(t)i(t) \\ &= \frac{1}{R}v^2(t) \\ &= Ri^2(t) \end{aligned} \tag{2.2}$$

We can *normalize* the instantaneous power by setting $R = 1$ which yields the *instantaneous normalized power*:

$$\begin{aligned} p(t) &= v^2(t) \\ &= i^2(t) \\ &= x^2(t) \end{aligned} \tag{2.3}$$

Since we have lost the dependence on resistance, $x(t)$ can be either the voltage or the current (or any other signal for that matter). Thus we prefer to use normalized power so that we don't need to specify resistances.

2.2.2 Total Normalized Energy

The instantaneous power tells us how much energy there is per second (recall that 1 Watt = 1 Joule per second). If we integrate the instantaneous power over a certain amount of time, then we will know how much energy there is in the signal over that time window. To compute the *total normalized energy* of the signal, simply integrate the instantaneous normalized power over all time:

$$\begin{aligned} E &= \lim_{T\to\infty}\int_{-T}^{T}|x(t)|^2dt \\ &= \int_{-\infty}^{\infty}|x(t)|^2dt \end{aligned} \tag{2.4}$$

Note that the magnitude operator $|\cdot|$ is there just in case $x(t)$ is a complex-valued signal [in this book $x(t)$ will usually be real valued and thus $|x(t)|^2 = x^2(t)$]. Unless otherwise specified, the term *energy* refers to the *total normalized energy* as defined by (2.4).

2.2.3 Average Normalized Power

While integrating the instantaneous power over a very wide time window gives the total amount of energy in the signal, dividing the energy by the width of the window gives the *average normalized power* of the signal:

$$P = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x(t)|^2dt \tag{2.5}$$

Unless the signal is periodic (see below), we need to remember to keep the limit operator in the equation. Unless otherwise specified, the term *power* refers to the *average normalized power* as defined by (2.5). Because of the limit operator, it is hard to compute (2.5) for an arbitrary signal ... but there is an exception ...

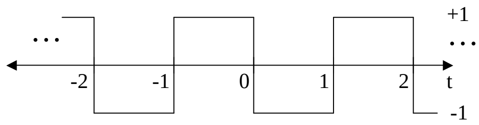
2.2.4 Power of a Periodic Signal

If the signal is periodic, the power is easy to find:

$$P = \frac{1}{T_o}\int_{t_o}^{t_o+T_o}|x(t)|^2dt \tag{2.6}$$

For any value of t_o. Note that you can do this integration over any period of the signal $x(t)$ by picking the value of t_o that is most convenient.

Example: Find the *power* of the following signal:



2.2.5 Power and Energy Signals

A signal is a *power signal* if it has finite nonzero power, $0 < P < \infty$ and is an *energy signal* if it has finite nonzero energy $0 < E < \infty$. A signal can sometimes be neither an energy or power signal, but can never be both.

Example: Classify the following as energy signal, power signal, or neither:

1. $x(t) = \Pi(t)$

2. $x(t) = \sin(t)$

3. $x(t) = e^{-t}u(t)$

4. $x(t) = tu(t)$

5. $x(t) = 0$

6. Any finite-valued periodic signal except $x(t) = 0$.

A few more questions to consider:

1. What is the power of an energy signal?

2. What is the energy of a power signal?

2.3 Complex Exponentials

When we take the Fourier Series or Fourier Transform of a signal, the result is a complex-valued sequence (the F.S. coefficients) or signal (the F.T. results in a function of frequency). Thus it is worthwhile to spend a few minutes reviewing the mathematics of complex numbers.

A complex number z can be represented in *rectangular* or *Cartesian* form:

$$z = x + jy \tag{2.7}$$

where x and y are each real-valued numbers and $j = \sqrt{-1}$.

Alternatively, z can be represented in *polar* form:

$$z = re^{j\theta} \tag{2.8}$$

where $e^{j\theta}$ is a *complex exponential* and r is a real-valued number. We call r the *magnitude* and θ the *phase*. Relating rectangular to polar coordinates:

$$x = r\cos\theta \tag{2.9}$$

$$y = r\sin\theta \tag{2.10}$$

$$r = \sqrt{x^2 + y^2} \tag{2.11}$$

$$\theta = \angle z \tag{2.12}$$

2.3.1 Euler's Equation

First, equate (2.7) and (2.8),

$$re^{j\theta} = x + jy. \tag{2.13}$$

Next, substitute (2.9) and (2.10),

$$re^{j\theta} = r\cos\theta + jr\sin\theta \tag{2.14}$$

Finally, divide both sides by r,

$$e^{j\theta} = \cos\theta + j\sin\theta. \tag{2.15}$$

The above expression is known as *Euler's equation.*

From Euler's equation we see that:

$$\Re\{e^{j\theta}\} = \cos\theta \tag{2.16}$$

$$\Im\{e^{j\theta}\} = \sin\theta \tag{2.17}$$

What if the exponential is negative? Then:

$$\begin{aligned} e^{-j\theta} &= \cos(-\theta) + j\sin(-\theta) \\ &= \cos\theta - j\sin\theta \end{aligned} \tag{2.18}$$

since cos is an even function and sin is an odd function.

We can represent $\cos\theta$ and $\sin\theta$ in terms of complex exponentials:

$$\begin{aligned} \cos\theta &= \frac{1}{2}\left(e^{j\theta}+e^{-j\theta}\right) && (2.19)\\ \sin\theta &= \frac{1}{2j}\left(e^{j\theta}-e^{-j\theta}\right) && (2.20) \end{aligned}$$

Proof:

2.3.2 Complex Conjugates

If $z = x + jy = re^{j\theta}$, then the *complex conjugate* of z is $z^* = x - jy = re^{-j\theta}$.

What happens when we add complex conjugates? Consider $z + z^*$

What if we multiply complex conjugates? Consider $(z)(z^*)$

2.4 Rotating Phasors

One of the most basic periodic signals is the sinusoidal signal

$$x(t) = r\cos(\omega_0 t + \theta), \tag{2.21}$$

where $\omega_0 = 2\pi f_o$ is the *angular frequency*, r is the *magnitude* and θ is the *phase* of the sinusoidal signal.

Another way to represent this sinusoid is as

$$\begin{aligned} x(t) &= r\Re\left\{e^{j(\omega_0 t+\theta)}\right\}\\ &= r\Re\left\{e^{j\theta}e^{j\omega_0 t}\right\}\\ &= \Re\left\{re^{j\theta}e^{j\omega_0 t}\right\}\\ &= \Re\left\{ze^{j\omega_0 t}\right\}, && (2.22) \end{aligned}$$

where $z = re^{j\theta}$. The quantity $ze^{j\omega_0 t}$ is called a *rotating phasor*, or just *phasor* for short. We can visualize a phasor as being a vector of length r whose tip moves in a circle about the origin of the complex plane. The original sinusoid $x(t)$ is then the projection of the phasor onto the real-axis. See the applet on the course webpage.

There is another way to use phasors to represent the sinusoidal signal without needing the $\Re\{\cdot\}$ operator. The idea is to use the property of complex conjugates that $z + z^* = 2\Re\{z\}$. This implies that

$$\Re\{z\} = \frac{z + z^*}{2}. \tag{2.23}$$

We can apply this property to the phasor representation of our sinusoidal signal:

$$\begin{aligned} x(t) &= \Re\left\{ze^{j\omega_0 t}\right\} \\ &= \frac{1}{2}\left[\left(ze^{j\omega_0 t}\right) + \left(ze^{j\omega_0 t}\right)^*\right] \\ &= \frac{1}{2}ze^{j\omega_0 t} + \frac{1}{2}z^* e^{-j\omega_0 t} \\ &= ae^{j\omega_0 t} + a^* e^{-j\omega_0 t}, \end{aligned} \tag{2.24}$$

where $a = z/2$. What (2.24) tells us is that we can represent a sinusoid as the sum of two phasors rotating at the same speed, but in opposite directions.

2.5 Sum of Two Sinusoids

Suppose we have the sum of two sinusoids, one at twice the frequency of the other

$$x(t) = r_1 \cos(\omega_1 t + \theta_1) + r_2 \cos(\omega_2 t + \theta_2), \tag{2.25}$$

where $\omega_1 = \omega_0$ and $\omega_2 = 2\omega_0$. Note that $x(t)$ is periodic with an angular frequency of ω_0. From (2.24), we can represent the signal as

$$\begin{aligned} x(t) &= a_1 e^{j\omega_1 t} + a_1^* e^{-j\omega_1 t} + a_2 e^{j\omega_2 t} + a_2^* e^{-j\omega_2 t} \\ &= a_1 e^{j\omega_0 t} + a_1^* e^{-j\omega_0 t} + a_2 e^{j2\omega_0 t} + a_2^* e^{-j2\omega_0 t} \qquad (2.26) \end{aligned}$$

$$\tag{2.27}$$

where $a_k = \frac{1}{2} r_k e^{j\theta_k}$, which implies that $|a_k| = r_k/2$ and $\angle a_k = \theta_k$.

If we let

$$\begin{aligned} a_{-1} &= a_1^* \\ a_{-2} &= a_2^* \\ a_0 &= 0 \end{aligned} \tag{2.28}$$

then we can represent $x(t)$ more compactly as

$$x(t) = \sum_{k=-2}^{k=+2} a_k e^{jk\omega_o t} \tag{2.29}$$

2.6 Fourier Series

Now suppose that we have a signal which is the sum of more than just two sinusoids, with each sinusoid having an angular frequency which is an integer multiple of ω_0. Furthermore, the signal could have a *DC component*, which is represented by a_0. Then the signal can be represented as

$$x(t) = a_0 + \sum_{k=1}^{\infty} r_k \cos(k\omega_0 t + \theta_k) \tag{2.30}$$

$$= a_0 + \sum_{k=1}^{\infty} \Re\left\{ z_k e^{jk\omega_0 t} \right\}, \tag{2.31}$$

where $z_k = r_k e^{j\theta_k}$. We can represent this signal as

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_o t}, \tag{2.32}$$

where again $a_k = \frac{1}{2} r_k e^{j\theta_k}$ and $a_{-k} = a_k^*$.

Theorem: Every periodic signal $x(t)$ can be represented as a sum of weighted complex exponentials in the form given by equation (2.32). Furthermore, the coefficients a_k can be computed using

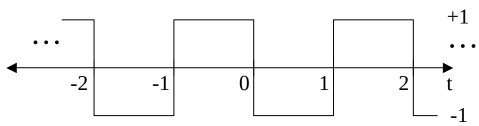
$$a_k = \frac{1}{T_o} \int_{t_o}^{t_o+T_o} x(t) e^{-jk\omega_o t} dt \tag{2.33}$$

Some notes:

- $\omega_o = 2\pi f_o = 2\pi/T_o$ is the fundamental angular frequency.
- The integral in (2.33) can be performed over any period of $x(t)$.
- The Fourier Series coefficients a_k are, in general, complex numbers.
- When $k > 0$, a_k is also called the *kth harmonic component*.
- Since $a_{-k} = a_k^*$, $|a_k|$ and $\Re\{a_k\}$ are even functions of k, while $\angle a_k$ and $\Im\{a_k\}$ are odd functions of k.
- a_o is the DC component, and it is real-valued.
- You should compute a_o as a separate case to avoid a divide by zero problem.

2.6.1 An Example

Find the complex Fourier Series representation of the following square wave.



2.7 Magnitude & Phase Spectra

We can depict the Fourier Series graphically by sketching the magnitude and phase of the complex coefficients a_k as a function of frequency. The Fourier Series tells us that the signal can be represented by a complex exponential located every kf_o Hz, and that this exponential will be weighted by amount a_k. The magnitude spectra just represents the magnitudes of the coefficients $A_k = |a_k|$ with lines of height A_k located at kf_o Hz. Likewise, the phase spectral shows the angle of the coefficients $\theta_k = \angle a_k$. *Sketch the magnitude and phase spectra for the example from the last page.*

2.7.1 Trig Form of the Fourier Series

From (2.31), we can represent any periodic signal as

$$x(t) \quad = \quad a_0 + \sum_{k=1}^{\infty} \Re\left\{ z_k e^{jk\omega_0 t} \right\}. \tag{2.34}$$

Let $b_k = \Re\{z_k\}$ and $c_k = -\Im\{z_k\}$, which implies that $z_k = b_k - jc_k$. Since $e^{jk\omega_0 t} = \cos(k\omega_0 t) + j\sin(k\omega_0 t)$, we have

$$\begin{aligned} x(t) &= a_0 + \sum_{k=1}^{\infty} \Re\left\{ [b_k - jc_k] \left[\cos(k\omega_0 t) + j \sin(k\omega_0 t) \right] \right\} && (2.35) \\ &= a_0 + \sum_{k=1}^{\infty} \Re\left\{ [b_k \cos(k\omega_0 t) + c_k \sin(k\omega_0 t)] \right\} && (2.36) \end{aligned}$$

The above is the *trig form* of the Fourier Series.

How are the coefficients b_k and c_k found? Recall that $a_k = \frac{1}{2} z_k$, so we can compute $z_k = 2a_k$,

$$\begin{aligned} z_k &= 2a_k \\ &= \frac{2}{T_o} \int_{t_o}^{t_o+T_o} x(t) e^{-jk\omega_o t} dt \\ &= \frac{2}{T_o} \int_{t_o}^{t_o+T_o} x(t) \left[\cos(k\omega_0 t) - j \sin(k\omega_0 t) \right] \\ &= \frac{2}{T_o} \int_{t_o}^{t_o+T_o} x(t) \cos(k\omega_0 t dt - \frac{2j}{T_o} \int_{t_o}^{t_o+T_o} x(t) \sin(k\omega_0 t) dt && (2.37) \end{aligned}$$

Since $b_k = \Re\{z_k\}$, we get

$$b_k \quad = \quad \frac{2}{T_o} \int_{t_o}^{t_o+T_o} x(t) \cos(k\omega_0 t dt \tag{2.38}$$

and since $c_k = -\Im\{z_k\}$, we get

$$c_k \quad = \quad \frac{2}{T_o}\int_{t_o}^{t_o+T_o} x(t)\sin(k\omega_0 t)dt \tag{2.39}$$

Note that the preferred form for the Fourier Series is the complex exponential Fourier Series given by Equation (2.32). So you might wonder why we bother with the trig form? Well, it turns out for some more complicated functions, it is more computationally efficient to first find the trig form coefficients b_k and c_k, and then use those coefficients to determine a_k.

Question: How can you find a_k from b_k and c_k?

2.8 Parseval's Theorem

Recall that the power of a periodic signal is found as:

$$P \quad = \quad \frac{1}{T_o}\int_{t_o}^{t_o+T_o} |x(t)|^2 dt \tag{2.40}$$

Since $|x(t)|^2 = x(t)x^*(t)$ this can be rewritten as:

$$P \quad = \quad \frac{1}{T_o}\int_{t_o}^{t_o+T_o} x(t)x^*(t)dt \tag{2.41}$$

Replace $x^*(t)$ with its complex F.S. representation:

$$\begin{aligned} P &= \frac{1}{T_o}\int_{t_o}^{t_o+T_o} x(t)\left(\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_o t}\right)^* dt \\ &= \frac{1}{T_o}\int_{t_o}^{t_o+T_o} x(t)\sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_o t}dt \end{aligned} \tag{2.42}$$

Pull the summation to the outside and rearrange some terms:

$$\begin{aligned} P &= \sum_{k=-\infty}^{\infty} a_k^*\left(\frac{1}{T_o}\int_{t_o}^{t_o+T_o} x(t)e^{-jk\omega_o t}dt\right) \\ &= \sum_{k=-\infty}^{\infty} a_k^* a_k \\ &= \sum_{k=-\infty}^{\infty} |a_k|^2 \end{aligned} \tag{2.43}$$

We see from the above equation that we can compute the power of the signal directly from the Fourier Series coefficients.

Parseval's Theorem: The power of a periodic signal $x(t)$ can be computed from its Fourier Series coefficients using:

$$
\begin{aligned}
P &= \sum_{k=-\infty}^{\infty} |a_k|^2 \\
&= a_o^2 + 2\sum_{k=1}^{\infty} |a_k|^2
\end{aligned} \qquad (2.44)
$$

Example: Use Parseval's Theorem to compute the power of the square wave from Section 2.6.1.

2.9 Exercises

1. For periodic signal (a) shown on the next page

- **(a)** Compute the power in the *time domain* (i.e. using equation 2.6).
- **(b)** Calculate the complex Fourier Series coefficients a_k. Try to get your answer into its simplest form.
- **(c)** Sketch the magnitude spectra $|a_k|$ for $|k| \leq 5$.
- **(d)** Use Parseval's theorem to compute the power from the Fourier Series coefficients. Compare your answer to that of part (a).

$$\text{Hint:} \quad \sum_{n=1}^{n=\infty} \frac{1}{n^2} \quad = \quad \frac{\pi^2}{6}$$

2. Repeat 1 for periodic signal (b).

3. Repeat 1.a, 1.b, and 1.c for periodic signal (c). *Hint: The book-keeping will be easier if you first compute the trig form coefficients b_k and c_k, and then use these to determine a_k.*

4. Repeat 1.b, and 1.c for periodic signal (d).

5. A periodic signal has the following complex Fourier Series coefficients:

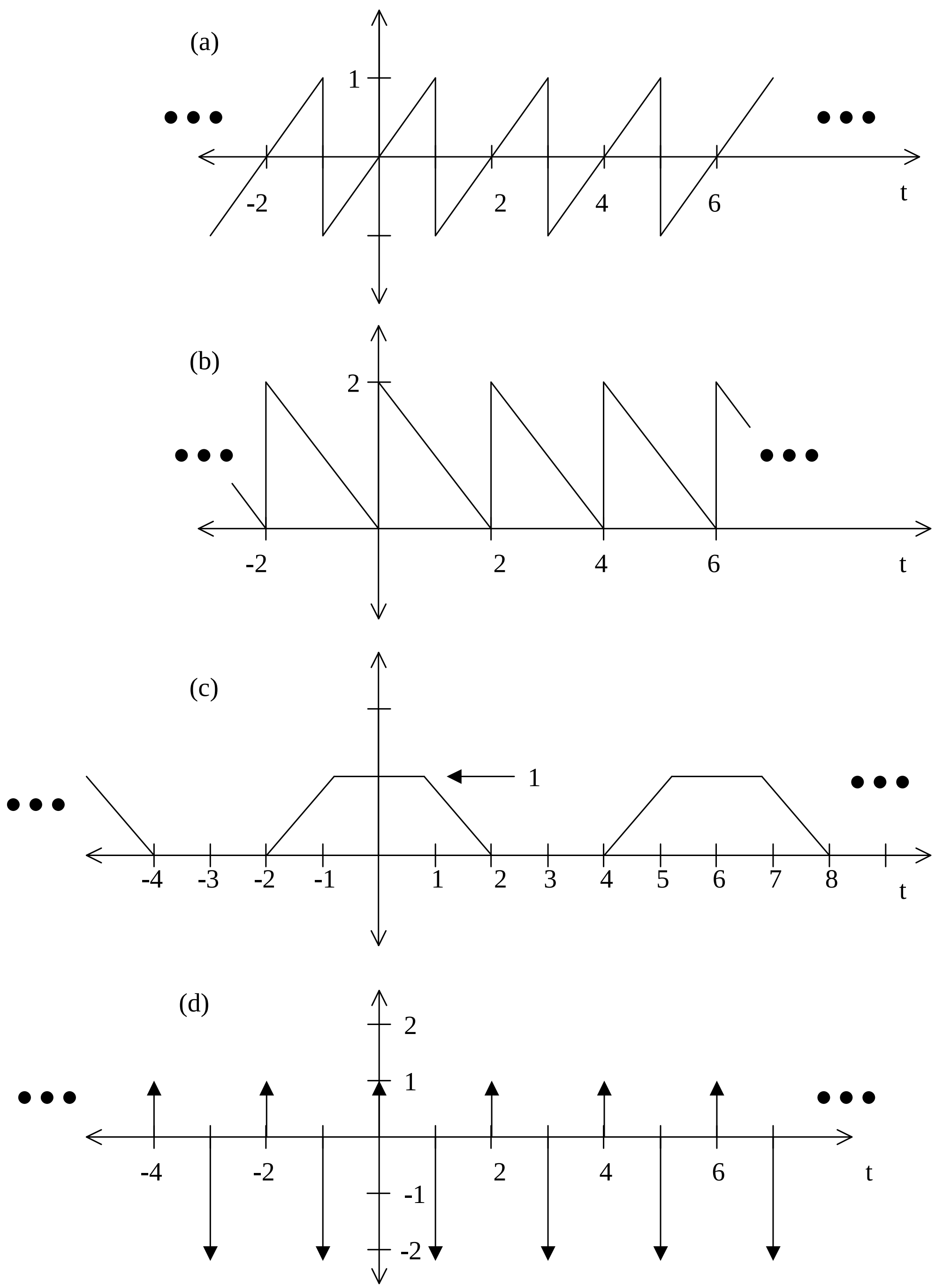
$$a_k \quad = \quad \begin{cases} 1 & \text{for } k = 0 \\ |k|e^{-jk\pi/2} & \text{for } 1 \leq |k| \leq 3 \\ 0 & \text{for } |k| > 3 \end{cases}$$

Compute the power of this signal. Give a numerical answer.

6. A periodic signal $x(t)$ can be represented as:

$$x(t) \quad = \quad \sum_{k=-N}^{N} k e^{jk2\pi t}$$

Determine the maximum value of the integer N such that the power of $x(t)$ does not exceed 20 Watts.



(a)
1
-2
2
4
6
t
(b)
2
-2
2
4
6
t
(c)
1
-4
-3
-2
-1
1
2
3
4
5
6
7
8
t
(d)
2
1
-4
-2
2
4
6
t
-1
-2

Chapter 3

The Fourier Transform

3.1 Definition of the Fourier Transform

If the signal $x(t)$ is periodic, we can use the Fourier Series to obtain a frequency domain representation. But what if $x(t)$ is *aperiodic*? The key is to think of an aperiodic signal as being a periodic one with a very large period. In other words, an aperiodic signal is merely a periodic one with fundamental period $T_o \to \infty$.

When the period gets large, a few things occur:

- The limit of integration in (2.33) used to form the coefficients goes from $-\infty$ to ∞.
- When we plot the magnitude spectra, we get a line every $f_o = 1/T_o$ Hz (this is why the magnitude spectra is sometimes called a line spectra). As $T_o \to \infty$ these lines turn into a continuous function of frequency.
- Because there are no longer discrete lines associated with particular frequencies, the summation in the Fourier Series representation (2.32) becomes an integral.

All these observations are captured in the *Fourier Transform*. The transform itself is similar to the equation for generating Fourier Series coefficients, and is defined as follows:

$$X(w) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{3.1}$$

The main difference is that we are now generating a continuous function of angular frequency (rather than a sequence of discrete coefficients), the limits of the integral are infinite, and the $1/T_o$ term in front of the integral is no longer there (since it would cause the function to be zero for all frequencies).

The *inverse Fourier Transform* is similar to the equation that expresses the signal as a function of the Fourier Series coefficients, and is as follows:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \tag{3.2}$$

Note that the F.S. coefficients in (2.32) have been replaced with the Fourier Transform of the signal, which requires the summation to be replaced with an integral. The $1/(2\pi)$ term is a consequence of representing the transform in terms of angular frequency.

We can represent the Fourier Transform in terms of the true frequency (in Hz) rather than the angular frequency by using the following relations:

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \\ x(t) &= \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df \end{aligned} \tag{3.3}$$

Both of these definitions of the Fourier Transform are used in practice, which is sometimes a source of confusion. Although you should be familiar with both representations (angular frequency and true frequency), we will use the true frequency version (3.3) for the remainder of the text (since this is what is more commonly used in practice ... how often do you hear the broadcasting frequency of an FM radio station expressed in rad/sec?).

Some other notation that is used:

$$\begin{aligned} X(f) &= \mathcal{F}\{x(t)\} \\ x(t) &= \mathcal{F}^{-1}\{X(f)\} \\ x(t) &\Leftrightarrow X(f) \end{aligned} \tag{3.4}$$

3.2 Common F.T. Pairs and Properties

We will now derive several Fourier Transform pairs and several properties of the Fourier Transform. These pairs and properties are summarized in a table at the end of this book.

3.2.1 F.T. Pair: Rectangular Pulse

Example: Let $x(t) = \Pi(t/T)$. Find $X(f)$.

3.2.2 F.T. Pair: Complex Exponential

Example: Let $x(t) = e^{j\omega_o t} = e^{j2\pi f_o t}$. Find $X(f)$.

3.2.3 F.T. Property: Linearity

Theorem: If $x(t) \Leftrightarrow X(f)$ and $y(t) \Leftrightarrow Y(f)$, then

$$ax(t) + by(t) \quad \Leftrightarrow \quad aX(f) + bY(f), \tag{3.5}$$

where a and b are constants.

Proof:

3.2.4 F.T. Property: Periodic Signals

Let $x(t)$ be a periodic signal with complex exponential Fourier Series coefficients a_k. Find the F.T. of $x(t)$ in terms of its F.S. coefficients.

3.2.5 F.T. Pair: Train of Impulses

Consider the train of dirac delta functions

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_o) \tag{3.6}$$

Find $X(f)$.

3.2.6 F.T. Property: Time Shifting

Theorem: If $x(t) \Leftrightarrow X(f)$, then

$$x(t - t_o) \Leftrightarrow e^{-j2\pi f t_o} X(f), \tag{3.7}$$

where t_o is a constant time delay.
Proof:

3.2.7 F.T. Property: Time Scaling

Theorem: If $x(t) \Leftrightarrow X(f)$, then

$$x\left(at\right) \quad \Leftrightarrow \quad \frac{1}{|a|} X\left(\frac{f}{a}\right), \tag{3.8}$$

where a is a constant time-scaling factor.

Proof:

3.2.8 Example: Using F.T. Properties

Find the F.T. for the following signal:

$$x(t) = 3\Pi\left(\frac{t-2}{2}\right) + \Pi\left(\frac{t}{10}\right) \tag{3.9}$$

3.2.9 F.T. Pair: Delta Function

Find the F.T. of $x(t) = \delta(t)$.

3.2.10 F.T. Pair: Constant

Find the F.T. of $x(t) = K$, where K is a constant.

3.2.11 F.T. Property: Duality

Theorem: Let the Fourier Transform of $x(t)$ be $X(f)$. Then the Fourier Transform of $X(t)$ is $x(-f)$.

Proof:

3.2.12 F.T. Pair: Sinc

Find the F.T. of $x(t) = \mathrm{sinc}(2Wt)$ where W is the "bandwidth" of $x(t)$.

3.2.13 F.T. Pair: Cosine

Find the F.T. of $x(t) = \cos(\omega_o t)$.

3.2.14 F.T. Property: Differentiation in Time

Theorem: If $x(t) \Leftrightarrow X(f)$, then

$$\frac{d^n}{dt^n} x(t) \quad \Leftrightarrow \quad (j2\pi f)^n X(f) \tag{3.10}$$

Proof:

3.2.15 F.T. Pair: Sine

Find the F.T. of $x(t) = \sin(\omega_o t)$ by using the fact that

$$\sin(\omega_o t) \quad = \quad -\left(\frac{1}{\omega_o}\right) \frac{d}{dt} \cos(\omega_o t)$$

3.2.16 F.T. Property: Integration

Theorem: If $x(t) \Leftrightarrow X(f)$, then

$$\int_{-\infty}^{t} x(\lambda)d\lambda \quad \Leftrightarrow \quad \frac{1}{j2\pi f}X(f) + \frac{X(0)}{2}\delta(f) \tag{3.11}$$

Proof: Can be obtained by integration by parts when $X(0) = 0$. When $X(0) \neq 0$ then a limiting argument must be used.

3.2.17 F.T. Pair: Unit-step

Find the F.T. of $x(t) = u(t)$ by using the fact that

$$u(t) \quad = \quad \int_{-\infty}^{t} \delta(t)dt$$

3.2.18 F.T. Property: Convolution

Theorem: If $x(t) \Leftrightarrow X(f)$ and $y(t) \Leftrightarrow Y(f)$ then

$$x(t) * y(t) \quad \Leftrightarrow \quad X(f)Y(f) \tag{3.12}$$

Proof:

3.2.19 F.T. Pair: Triangular Pulse

Find the F.T. of $x(t) = \Lambda(t/T)$ using the fact that

$$\Pi(t) * \Pi(t) \quad = \quad \Lambda(t)$$

or, more generally

$$\Pi\left(\frac{t}{T}\right) * \Pi\left(\frac{t}{T}\right) \quad = \quad T\Lambda\left(\frac{t}{T}\right)$$

3.2.20 F.T. Pair: Train of pulses

Find the F.T. of the following train of pulses:

$$x(t) = \sum_{-\infty}^{\infty} \Pi\left(\frac{t-kT_o}{\tau}\right) \tag{3.13}$$

By using

$$\sum_{-\infty}^{\infty} \Pi\left(\frac{t-kT_o}{\tau}\right) = \Pi\left(\frac{t}{\tau}\right) * \sum_{-\infty}^{\infty} \delta\left(t-kT_o\right) \tag{3.14}$$

3.2.21 F.T. Property: Multiplication

Theorem: If $x(t) \Leftrightarrow X(f)$ and $Y(t) \Leftrightarrow Y(f)$ then

$$x(t)y(t) \Leftrightarrow X(f) * Y(f) \tag{3.15}$$

Proof: This is merely the dual of the convolution property.

3.2.22 F.T. Pair: Sinc-squared

Find the F.T. of $x(t) = sinc^2(2Wt)$

3.2.23 F.T. Property: Frequency Shifting

Theorem: If $x(t) \Leftrightarrow X(f)$, then

$$e^{j\omega_o t}x(t) \quad \Leftrightarrow \quad X(f - f_o) \tag{3.16}$$

where $\omega_o = 2\pi f_o$

Proof: This is merely the dual of the time-delay property.

Example: Let $x(t) = \text{sinc}(2t)$. Find the F.T. of $y(t) = e^{j\omega_o t}x(t)$ when $\omega_o = 4\pi$.

3.2.24 F.T. Pair: Decaying exponential

Find the F.T. of $x(t) = e^{-at}u(t)$

3.2.25 F.T. Property: Differentiation in Frequency

Theorem: If $x(t) \Leftrightarrow X(f)$, then

$$t^n x(t) \quad \Leftrightarrow \quad (-j2\pi)^{-n} \frac{d^n}{df^n} X(f) \tag{3.17}$$

Proof: This is merely the dual of the differentiation in time property.

3.2.26 F.T. Pair: $te^{-at}u(t)$

Find the F.T. of $x(t) = te^{-at}u(t)$

3.3 Exercises

1. Find the Fourier Transform of the function $x(t) = e^{-2|t|}$. Express your answer in terms of absolute frequency f (rather than angular frequency ω). For this problem, you should use the Fourier Transform integral [i.e. equation (3.3)].

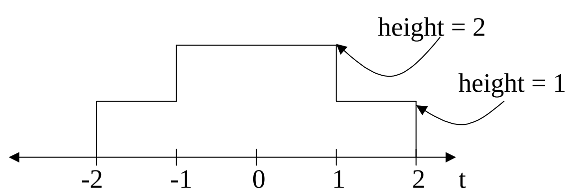
2. Find the Fourier transform (using the appropriate properties) of

$$x(t) = \sin(2\pi f_1 t)\cos(2\pi f_2 t)$$

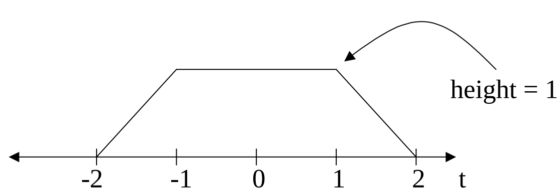
3. Given that $\mathcal{F}\{x(t)\} = X(f)$ find an expression for $\mathcal{F}\{x(a(t - T_d))\}$.

4. Using the properties and transform pairs derived in this chapter, find the Fourier Transform $X(f)$ of each of the following functions (express your answer in terms of f, simplify the expression as much as you can, and sketch your results for parts c, e, f, and g):

(a) The signal shown below.



(b) The signal shown below (hint: think of this as a big triangle with a smaller triangle subtracted from it).



(c) $x(t) = 2\text{sinc}(2t) + 4\text{sinc}(4t)$. *Sketch your result.*

(d) $x(t) = \delta\left(\frac{t-3}{2}\right)$

(e) $x(t) = \text{sinc}^2\left(\frac{t}{2}\right)$. *Sketch your result.*

(f) $x(t) = \frac{1}{j2\pi t} + \frac{1}{2}\delta(t)$. *Sketch your result.*

(g) $x(t) = e^{j6\pi t}\text{sinc}^2(t)$. *Sketch your result.*

(h) $x(t) = e^{-(t-3)}u(t-1)$

5. Consider the following two signals:

$$\begin{aligned} x_1(t) &= \Pi(t-1) \\ x_2(t) &= \Pi(t-2) \end{aligned}$$

(a) Find the Fourier Transform of each of $x_1(t)$ and $x_2(t)$ in terms of absolute frequency f.

(b) Find the product of the two Fourier Transforms $Y(f) = X_1(f)X_2(f)$.

(c) Compute the inverse Fourier Transform of your answer to (b), $y(t) = \mathcal{F}^{-1}\{Y(f)\}$. Sketch your result.

d How does your answer to (c) compare to the answer to problem 4(b) from chapter #1? Why?

6. The following signal

$$x(t) = 4\cos(2000\pi t) + 2\cos(5000\pi t)$$

is passed through a LTI system with impulse response:

$$h(t) = 2000\text{sinc}^2(2000t)$$

so that the output is $y(t) = x(t) * h(t)$.

1. Find and sketch $X(f) = \mathcal{F}\{x(t)\}$, the Fourier Transform of the input.
2. Find and sketch $H(f) = \mathcal{F}\{h(t)\}$, the frequency response of the system.
3. Find and sketch $Y(f)$, the Fourier Transform of the output.
4. Find a simplified expression for $y(t) = \mathcal{F}^{-1}\{Y(f)\}$.

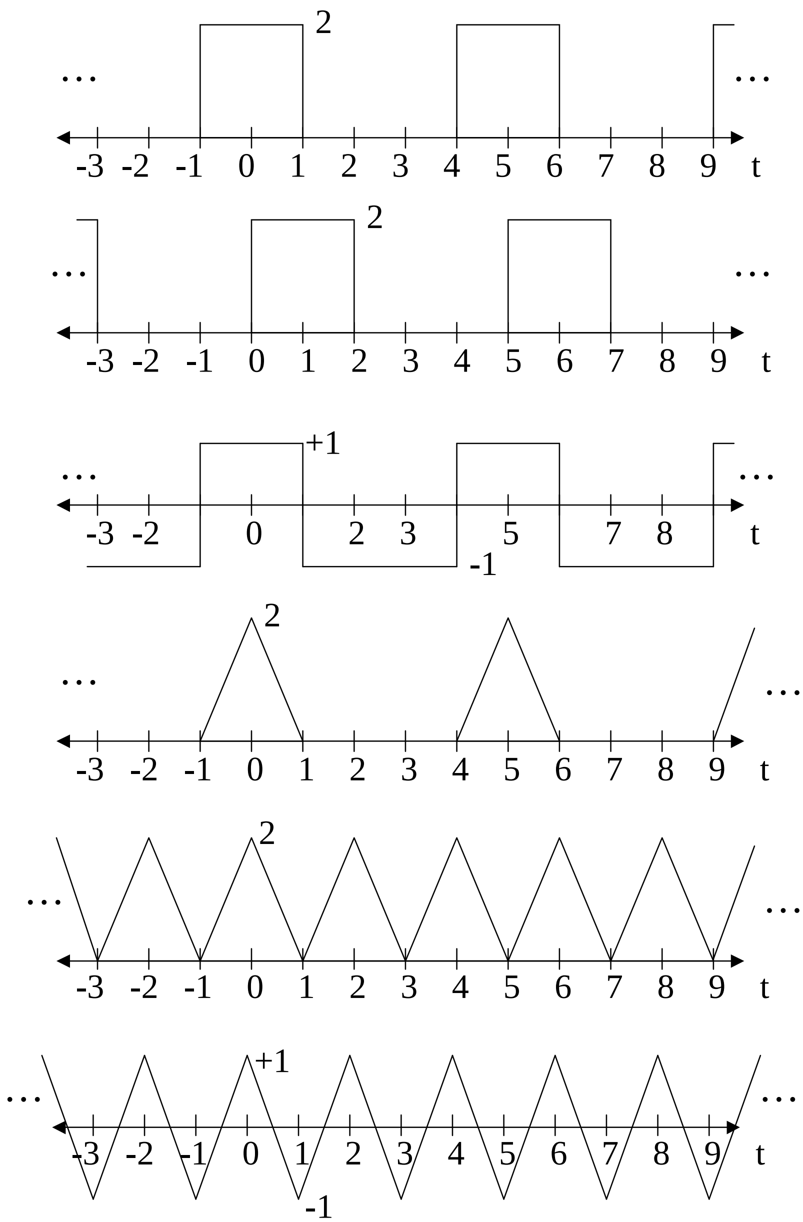
7. For each of the periodic signals considered in problems 1-4 in chapter #2, determine the Fourier Transform.

8. Find the Fourier Transform for each of the following signals. Simplify your answer as much as you can.

$$\begin{aligned} x_1(t) &= \cos\left(2000\pi t + \frac{\pi}{4}\right) \\ x_2(t) &= \cos(2000\pi t)\, u(t) \\ x_3(t) &= t^2 e^{-2t} u(t) \\ x_4(t) &= t\text{sinc}^2(t) \\ x_5(t) &= \int_{-\infty}^{t} \Lambda(\lambda) d\lambda \\ x_6(t) &= \frac{1}{(2 + j2\pi t)^2} \\ x_7(t) &= \frac{1}{1 + \pi^2 t^2} \end{aligned}$$

Hint: For the last signal, you may want to leverage the solution to problem #1.

9. Find the Fourier Transform of each of the periodic signals shown below:



Chapter 4

Filtering

4.1 Lowpass Signals and Filters

4.1.1 Ideal Lowpass Filter

An **ideal lowpass filter** $h_{lpf}(t)$ passes (with no attenuation or phase shift) all frequency components with absolute frequency $|f| < W$, where W is the *bandwidth* of the filter, and attenuates (completely) all frequency components with absolute frequency $|f| > W$.

Q: From the convolution theorem, what must the frequency response $H_{lpf}(f)$ be?

Therefore, what is the impulse response $h_{lpf}(t)$?

4.1.2 Lowpass Signals

A **lowpass** or **baseband signal** is a signal that can be passed through a lowpass filter (with finite bandwidth W) without being distorted (i.e. its spectrum for all $|f| > W$ is zero). The minimum value of W for which the signal is passed undistorted is the **absolute bandwidth** of the signal.

Q: For each of the following signals, determine if it is a baseband signal, and if so find its bandwidth:

$$x(t) = \Pi(t)$$

$$x(t) = \Lambda(t)$$

$$x(t) = \text{sinc}(4000t)$$

$$x(t) = 1 + \cos(4000\pi t)$$

$$x(t) = e^{-t}u(t)$$

4.1.3 Practical Lowpass Filters

It should be noted that an ideal lowpass filter is not attainable in practice.

Q: Why? (Consider the impulse response).

A practical lowpass filter has three regions:

Passband: All frequencies below W_p are passed with minimal distortion. Note however, that the passband is not entirely flat. The *passband ripple* δ_1 is the tolerance of the ripple in the passband, i.e. in the passband, the frequency response satisfies $1 - \delta_1 \leq |H(f)| \leq 1 + \delta_1$. Additionally, the passband may change the phase of the signal, although the phase response of the filter will just be a linear function of frequency (which implies a constant time delay).

Stopband: Frequencies above W_s are almost completely attenuated (but not entirely). However, as with the passband, the stopband is not entirely flat. The *stopband ripple* δ_2 is the tolerance of the ripple in the stopband, i.e. in the stopband, the frequency response satisfies $|H(f)| \leq \delta_2$.

Transition band/region: The frequency response of the filter cannot have a sharp transition, and thus must gradually fall from $H(f) \approx 1$ at $f = W_p$ to $H(f) \approx 0$ at $f = W_s$. In the transition band, $1 - \delta_1 \geq |H(f)| \geq \delta_2$. The center of the transition band is W, which is called the *cutoff frequency.*

Classes of practical filters.

Butterworth Filter: Can only specify cutoff frequency W and the filter "order" N. In MATLAB, use `>>butter`.

Chebyshev Filter: Can also specify the passband ripple δ_1. In MATLAB, use `>>cheby1` or `>>cheby2`.

Elliptic (Cauer) Filter: Can specify both the passband ripple δ_1 and the stopband ripple δ_2. In MATLAB, use $>>$ `ellip`.

4.2 Highpass Signals and Filters

4.2.1 Ideal Highpass Filter

An **ideal highpass filter** $h_{hpf}(t)$ passes (with no attenuation or phase shift) *all* frequency components with absolute frequency $|f| > W$, where W is the **cutoff frequency** the filter, and attenuates (completely) all frequency components with absolute frequency $|f| < W$.

Q: From the convolution theorem, what must the frequency response $H_{hpf}(f)$ be?

Therefore, what is the impulse response $h_{hpf}(t)$?

4.2.2 Highpass Signals

A **highpass signal** is a signal that can be passed through a highpass filter (with finite nonzero cutoff frequency W) without being distorted (i.e. its spectrum for $|f| < W$ is zero). However, if the signal can also be a bandpass signal (defined below), it is not considered a highpass signal. The **absolute bandwidth** of highpass signals is infinite.

4.2.3 Practical Highpass Filters

Like practical lowpass filters, a practical highpass filter has a passband, stopband, and transition region. However, the stopband is lower in frequency than the passband, i.e $W_s < W_p$.

4.3 Bandpass Signals and Filters

4.3.1 Ideal Bandpass Filters

An **ideal bandpass filter** $h_{bpf}(t)$ passes (with no attenuation or phase shift) *all* frequency components with absolute frequency $W_1 < |f| < W_2$, where W_1 is the *lower* cutoff frequency the filter ad W_2 is the *upper* cutoff frequency, and it attenuates (completely) all frequency components with absolute frequency $|f| < W_1$ or $|f| > W_2$.

Q: From the convolution theorem, what must the frequency response $H_{bpf}(f)$ be?

Therefore, what is the corresponding impulse response?

4.3.2 Bandpass Signals

A **bandpass signal** is a signal that can be passed through a bandpass filter (with finite nonzero W_1 and W_2) without being distorted (i.e. its spectrum for $|f| < W_1$ and $|f| > W_2$ is zero). The **absolute bandwidth** of the bandpass signal is $W = W_2 - W_1$.

Q: For each of the following signals, determine if it is a bandpass signal, and if so find its bandwidth:

$$x(t) = \cos(10000\pi t)\mathrm{sinc}\,(1000t)$$

$$x(t) = 1 + \cos(4000\pi t)$$

$$x(t) = \cos(2000\pi t) + \cos(4000\pi t)$$

4.3.3 Ideal Bandpass Filters

Ideal bandpass filters have a passband, two stopbands, and two transition bands

4.4 Example: Putting It All Together

For each of the following signals, determine if it is a baseband, highpass, or bandpass signal, and if so, find its bandwidth:

$$x(t) = \Pi(1000t)$$

$$x(t) = \text{sinc}^2(2t)\cos(4\pi t)$$

$$x(t) = 1000\text{sinc}(1000t)$$

$$x(t) = 1000\cos(10000\pi t)\text{sinc}(1000t)$$

$$x(t) = 1 + \cos(4000\pi t)$$

$$x(t) = \cos(2000\pi t) + \cos(4000\pi t)$$

$$x(t) = \delta(t) - 1000\text{sinc}(1000t)$$

4.5 Exercises

1. Find and sketch the Fourier Transform for each of the following signals. **Classify** each as one of the following: (a) Lowpass, (b) Highpass, (c) Bandpass, or (d) None-of-the-above. Note that each signal should only belong to one category (the categories are mutually exclusive). In addition, state the **bandwidth** for each of these signals (even if infinite).

$$\begin{aligned}
x_1(t) &= \cos^2(1000\pi t) \\
x_2(t) &= \sum_{k=-\infty}^{\infty} \delta(t-k) \\
x_3(t) &= 1000t\text{sinc}^2(1000t) \\
x_4(t) &= \cos(2000\pi t)u(t) \\
x_5(t) &= 4000\text{sinc}^2(2000t)\cos\left(2\pi(10^6)t\right) \\
x_6(t) &= \delta(t) - 2000\text{sinc}(2000t) \\
x_7(t) &= \sin(1000\pi t) + \cos(2000\pi t) \\
x_8(t) &= 4sinc(2t)\cos(6\pi t)
\end{aligned}$$

2. The following baseband signal:

$$x(t) \quad = \quad 1 + 4\cos(2000\pi t) + 6\cos(4000\pi t) + 8\cos(6000\pi t)$$

is passed through an ideal filter with frequency response $h(t)$. **Find the output of the filter** when the filter is:

(a) An ideal lowpass filter with cutoff at 1500 Hz.

(b) An ideal highpass filter with cutoff at 2500 Hz.

(c) An ideal bandpass filter with passband between 1500 Hz and 2500 Hz.

3. Consider a linear time invariant (LTI) system with input $x(t)$ and impulse response $h(t)$. The output is $y(t) = x(t) * h(t)$.

(a) If $x(t) = \Pi(t)$ and $y(t) = 5\Lambda(t)$, what must $h(t)$ be?

(b) If $x(t) = \Pi(t)$ and $y(t) = \Pi(t-1)$, what must $h(t)$ be?

(c) If $x(t) = \Pi(t)$ and $y(t) = \Pi(t-1) + \Pi(t+1)$, what must $h(t)$ be?

(d) If $x(t) = \Pi(t)$ and $y(t) = \Pi\left(\frac{t}{2}\right)$, what must $h(t)$ be?

(e) Now assume that $x(t) = \text{sinc}(2000\pi t)$ and $h(t) = \text{sinc}(1000\pi t)$. Find and sketch $X(f)$, $H(f)$, and $Y(f)$. Find $y(t)$.

4. Let the input $x(t)$ of an ideal bandpass filter be:

$$x(t) = 1 + \cos(20\pi t) + \cos(40\pi t)$$

Determine and sketch the frequency response $H(f)$ for the filter such that the output $y(t) = x(t) * h(t)$ is

$$y(t) = \cos(20\pi t)$$

Chapter 5

Sampling

5.1 Sampling

Ideal or *impulse* sampling is the process of multiplying an arbitrary function $x(t)$ by a train of impulses, i.e.

$$x_s(t) \quad = \quad x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT_s) \tag{5.1}$$

where T_s is called the *sample period*, and $f_s = 1/T_s$ is the *sample rate* or *sample frequency*. Notice that sampling turns a *continuous-time* signal into a *discrete-time* signal.

Example: Let $x(t) = \cos(2\pi t)$ and $f_s = 8$. Sketch $x_s(t)$ for $0 \leq t \leq 2$

5.1.1 Fourier Transform of a Sampled Signal

Let $x(t) \Leftrightarrow X(f)$. Then the F.T. of the sampled signal $x_s(t)$ is

$$X_s(f) \quad \Leftrightarrow \quad f_s \sum_{k=-\infty}^{\infty} X(f - kf_s) \tag{5.2}$$

Proof: Find $X_s(f)$ as a function of $X(f)$.

5.1.2 Example: F.T. of Sampled Signal

Let $x(t) = 1000\text{sinc}^2(1000t)$. Sketch $X_s(f)$ for the following sample rates:

1. $f_s = 4000$

2. $f_s = 2000$

3. $f_s = 1500$

4. $f_s = 1000$

5.2 Nyquist Sampling Theorem

5.2.1 Minimum f_s

Question: If a baseband signal with bandwidth W is sampled at a rate of f_s Hz, what is the minimum value of f_s for which the spectral copies do not overlap?

5.2.2 Recovering $x(t)$ from $x_s(t)$

Question: Assume that a baseband signal is sampled with sufficiently high f_s. How can the original signal $x(t)$ be recovered from the sampled signal $x_s(t)$?

5.2.3 Nyquist Sampling Theorem

The **Nyquist Sampling Theorem** states that if a signal $x(t)$ has a finite bandwidth W, then it can be *uniquely* represented by samples taken at intervals of T_s seconds, where $f_s = 1/T_s \geq 2W$. The value $2W$ is called the **Nyquist rate**.

5.2.4 Digital-to-analog Conversion

A corollary to the above theorem is that if a baseband signal $x(t)$ with bandwidth W is sampled at a rate $f_s \geq 2W$, then the original signal $x(t)$ can be recovered by first passing the sampled signal $x_s(t)$ through an ideal lowpass filter with cutoff between W and $f_s - W$ and then multiplying the filter output by a factor of T_s.

Thus the frequency response of an ideal digital-to-analog converter (DAC) is:

$$H(f) \quad = \quad T_s \Pi\left(\frac{f}{f_s}\right) \tag{5.3}$$

The $\Pi(\cdot)$ term serves the purpose of filtering out all the spectral copies except the original ($k = 0$) copy, while the T_s term is required to make sure the reconstructed signal has the right amplitude [notice that the T_s term will cancel the f_s term in Equation (5.2) since $f_s = 1/T_s$).

5.2.5 Aliasing

If the signal is sampled at $f_s < 2W$ then frequency components above W will be "folded" down to a lower frequency after the DAC process. When a high frequency component is translated to a lower frequency component, this process is called **aliasing**.

5.2.6 Example

Given the following signal:

$$x(t) = 1 + 2\cos(400\pi t) + 2\cos(800\pi t) \tag{5.4}$$

1. What is the minimum value of f_s for which the original signal can be recovered from the sampled signal without aliasing?

2. If the signal is sampled at $f_s = 700$ Hz, then what would the signal at the output of an ideal DAC be?

3. If the signal is sampled at $f_s = 600$ Hz, then what would the signal at the output of an ideal DAC be?

5.2.7 Anti-aliasing Filter

Many real-world signals are not bandlimited (i.e. they don't have a finite bandwidth W). For such signals, aliasing can be prevented by passing the signal through a lowpass filter prior to sampling. Such a filter is called an *anti-aliasing filter* and should have a cutoff of $f_s/2$.

5.3 Exercises

1. For each of the following signals, **determine the smallest sampling rate** such that the original signal can be reconstructed from the sampled signal without any aliasing:

$$
\begin{aligned}
x_1(t) &= 2000\text{sinc}^2(2000t) \\
x_2(t) &= \sin^2(10,000\pi t) \\
x_3(t) &= 1000\text{sinc}\left(1000(t-1)\right)
\end{aligned}
$$

2. The following signal:

$$
x(t) = 3 + 4\cos(2000\pi t) + 2\cos(4000\pi t)
$$

is sampled at a rate of f_s samples/second. **Find and sketch the Fourier transform of the sampled signal** $x_s(t)$ **when**:

(a) $f_s = 10,000$

(b) $f_s = 5,000$

(c) $f_s = 4,000$

(d) $f_s = 3,000$

Show that when $f_s = 3,000$ the Fourier Transform of the sampled signal can be expressed in the form:

$$
X_s(f) = f_1 \sum_{k=-\infty}^{\infty} \delta(f - kf_2)
$$

and **determine the value of the constants** f_1 **and** f_2.

3. Consider the following analog signal:

$$
x(t) = \cos^2(200\pi t)
$$

Answer the following:

(a) What is the minimum rate that this signal may be sampled such that it can be recovered without aliasing?

For the remaining questions, assume that this signal is sampled at $f_s = 500$ Hz.

(b) Sketch the Fourier Transform of the sampled signal. Make sure you label the height and location of each impulse in your sketch.

(c) Sketch the frequency response of a filter that could be used to recover $x(t)$ from the sampled signal.

(d) Assume that the sampled signal is passed through an ideal bandpass filter with pass-band between 400 and 600 Hz. Find the output of the filter (as a function of time, not frequency).

4. So far, we only considered sampling lowpass signals. However, some bandpass signals can also be sampled and completely recovered, provided that they are sampled at a sufficiently high rate. Consider the following bandpass signal:

$$x(t) \quad = \quad 2\cos(800\pi t) + 2\cos(1000\pi t)$$

What is the **bandwidth** W of this signal? Suppose that the signal is sampled at exactly the Nyquist rate, $f_s = 2W$. **Find and sketch the Fourier Transform of the sampled signal** $x_s(t)$ over the range $-600 \leq f \leq 600$. By looking at your sketch, suggest a way to recover the original signal from the sampled signal. In particular, **specify the frequency response of an ideal digital-to-analog converter** for this system.

Chapter 6

Communications

6.1 Communication Systems

- *Communication systems* transport an *information bearing signal* or *message* from a *source* to a *destination* over a *communication channel*.

- Most information bearing signals are lowpass signals:

  1. Voice/music.
  2. Video.
  3. Data.

- Most communication channels act as bandpass filters:

  1. Wireless: cellular, microwave, satellite, infrared.
  2. Wired: coaxial cable, fiber-optics, twisted-pair (telephone lines, DSL).

- Thus, there is a mismatch between the source of information and the channel.

  – The source is a lowpass signal but the channel is a bandpass filter.

  – We know that lowpass signals cannot get through a bandpass filter.

  – What we need to do is translate the lowpass message signal to a bandpass signal suitable for delivery over the channel.

6.2 Modulation

Modulation is the process of translating a *lowpass* signal into a *bandpass* signal in such a way that the original lowpass signal can be recovered from the bandpass signal.

6.2.1 Types of Modulation

There are two main categories of modulation:

Linear: The lowpass (modulating) signal is multiplied by a high frequency *carrier* signal to form the modulated signal. Amplitude Modulation (AM) is linear.

Angle: The lowpass (modulating) signal is used to vary the phase or frequency of a high frequency carrier signal. Examples: Frequency Modulation (FM) and Phase Modulation (PM).

6.2.2 Simple Linear Modulation: DSB-SC

Let $x(t)$ be a lowpass signal with bandwidth W. If we multiply $x(t)$ by $c(t) = \cos(2\pi f_c t)$ we obtain a bandpass signal centered at f_c, which is called the *carrier frequency* or *center frequency.* Thus the modulated signal is:

$$x_m(t) = x(t)\cos(2\pi f_c t)$$

This type of modulation is called *Double Sideband Suppressed Carrier* (DSB-SC). Sketch a diagram of a DSB-SC modulator here:

Example: Let $x(t) = 100\text{sinc}^2(100t)$ and the carrier frequency be $f_c = 1000$ Hz. Find the Fourier Transform of $x_m(t)$.

6.2.3 Modulation Theorem

In general, if $x(t) \Leftrightarrow X(f)$ then

$$x(t)\cos(2\pi f_c t) \quad \Leftrightarrow \quad \frac{1}{2}X(f-f_c)+\frac{1}{2}X(f+f_c) \tag{6.1}$$

Proof:

6.2.4 Minimum value of f_c

In order to be able to recover $x(t)$ from $x_m(t)$, then f_c must be a minimum value. What is this minimum value?

A (degenerate) example: Let $x(t) = 100\text{sinc}^2(100t)$ and the carrier frequency be $f_c = 50$ Hz. Find the Fourier Transform of $x_m(t)$.

6.2.5 Demodulation

Given $x_m(t)$ (and assuming $f_c > W$) then how can $x(t)$ be recovered (i.e. design a demodulator)?

6.3 DSB-LC

6.3.1 Motivation

A problem with DSB-SC is that it requires a *coherent* (or *synchronous*) receiver. With coherent reception, the phase of the receiver's oscillator (i.e. the unit that generates $\cos(\omega_c t)$) must have the same phase as the carrier of the received signal. In practice, this can be implemented with a circuit called a *Phase Locked Loop* (PLL). However, PLLs are expensive to build. Thus, we might like to design a system that uses a signal that can be detected *noncoherently*, i.e. without needing a phase-locked oscillator. Cheap receivers are essential in *broadcasting* systems, i.e. systems with thousands of receivers for every transmitter.

6.3.2 Definition of DSB-LC

A circuit called an *envelope detector* can be used to noncoherently detect a signal. However, for an envelope detector to function properly, the modulating signal must be positive. We will see shortly that this is because the envelope detector rectifies the negative parts of the modulating signal. We can guarantee that the modulating signal is positive by adding in a DC offset A to it, i.e. use $A+x(t)$ instead of just $x(t)$. Thus the modulated signal is:

$$x_m(t) = (A + x(t))\cos(\omega_c t) \tag{6.2}$$

This type of signal is called **Double Sideband Large Carrier** (DSB-LC). Because this is the type of modulation used by AM broadcast stations, it is often just called *AM.*

Sketch a block diagram of a DSB-LC transmitter:

What does a DSB-LC signal look like in the frequency domain?

6.3.3 Envelope Detector

An **envelope detector** is a simple circuit capable of following the *positive envelope* of the received signal. Sketch the schematic of an envelope detector here:

In order to understand the operation of an envelope detector, it is helpful to sketch a typical DSB-LC signal (in the time domain) and then show what the output of the detector would look like.

6.3.4 Modulation index

If the constant A is not high enough, then the modulating signal is not guaranteed to be positive. This causes the signal at the output of the envelope detector to be distorted, due to rectification of the negative parts. Let's define a positive constant called the **modulation index**:

$$\begin{aligned} m &= \frac{K}{A} && (6.3)\\ &= \frac{1}{A}\max|x(t)| && (6.4)\end{aligned}$$

We can then rewrite our modulated signal as:

$$\begin{aligned} x_m &= (A + x(t))\cos(\omega_c t) && (6.5)\\ &= A\left(1 + \frac{1}{A}x(t)\right) && (6.6)\\ &= A\left(1 + \left(\frac{K}{A}\right)\left(\frac{x(t)}{K}\right)\right) && (6.7)\\ &= A\left(1 + m\tilde{x}(t)\right) && (6.8)\end{aligned}$$

where $\tilde{x}(t) = x(t)/K$ is the normalized[1] version of $x(t)$.

From this equation, we see that the modulating signal is negative whenever $m\tilde{x}(t) < -1$. Since $\max|\tilde{x}(t)| = 1$, we assume that $\min\tilde{x}(t) = -1$. This implies that $m \leq 1$ in order to guarantee that the modulating signal is positive. When $m > 1$, we say that the signal is **overmodulated**, and an envelope detector cannot be used to recover the original message $x(t)$ without distortion.

[1] Normalized means that it is scaled so that $\max|\tilde{x}(t)| = 1$.

6.4 Single Sideband

The two forms of DSB modulation that we have studied (DSB-SC and DSB-LC) get their name because both the positive and negative frequency components of the lowpass message signal are translated to a higher frequency. Thus there are two *sidebands*: a *lower* sideband that corresponds to the negative frequency components of the message signal, and an *upper* sideband that corresponds to the positive frequency components of the message. Thus, when we use DSB modulation, the bandwidth of the modulated signal is twice that of the message signal.

However, this doubling of bandwidth is wasteful. The two sidebands are redundant, in the sense that one is the mirror image of the other. Thus, at least in theory, we really only need to transmit either the upper or lower sideband. This is the principle behind **single sideband modulation** (SSB). With SSB, only one or the other sideband (upper or lower) is transmitted. Note that the transmitter and receiver circuitry for SSB is much more complicated that for DSB. However, SSB is more bandwidth efficient than DSB. SSB is commonly used in amateur radio bands.

6.5 Comparison of Linear Modulation

For each of the following, rank the three forms of modulation DSB-SC, DSB-LC, or SSB from "most" to "least"

Bandwidth efficiency:

Power efficiency (transmitter battery life):

Receiver complexity (cost):

6.6 Angle Modulation

With angle modulation, the message is encoded into the phase of the carrier, i.e.

$$\begin{aligned} x_m(t) &= A\cos\left(\omega_c t + \theta_c(t)\right) \\ &= A\cos\left(\omega_c t + g[x(t)]\right) \end{aligned}$$

where the phase $\theta_c(t) = g[x(t)]$ is defined by a function $g(t)$ of the message $x(t)$. Depending on how $g(t)$ is defined, the modulation can be either *phase modulation* (PM) or *frequency modulation* (FM).

6.6.1 Phase Modulation

With PM,

$$\theta_c(t) = \theta_o + k_p x(t)$$

where k_p is the *phase sensitivity* and θ_o is the *initial phase*. Without loss of generality, we may assume that $\theta_o = 0$, in which case the phase of the carrier is proportional to the message signal. Note that PM is not currently widely used.

6.6.2 Frequency Modulation

With FM, the information signal is used to vary the carrier frequency about the center value f_c. Note that frequency is the derivative of phase. Thus if the modulated signal is expressed as:

$$x_m(t) = A\cos\theta(t)$$

then the phase $\theta(t)$ is related to the message by

$$\frac{d\theta(t)}{dt} = \omega_c + k_f x(t)$$

where k_f is the *frequency sensitivity*. Alternatively, we can think of phase as the integral of frequency and thus the modulated signal is:

$$x_m(t) = A\cos\left(\omega_c t + k_f \int_{-\infty}^{t} x(\lambda)d\lambda\right)$$

Due to the existence of inexpensive equipment to produce and receive this type of signal, FM is the most popular type of analog modulation. Examples: FM radio, AMPS (analog) cell phones.

Spectrum of FM: Because the signal is embedded in the phase of the carrier, finding the Fourier Transform of a FM signal with an arbitrary modulating signal is very difficult. However, we can find the F.T. if we assume that the modulating signal is sinusoidal, e.g. if $x(t) = \cos(\omega_m t)$. But this gets complicated and goes beyond the scope of this book.

6.7 Exercises

1. The following message signal

$$x(t) = 400\text{sinc}(200t)$$

is used to modulate a carrier. The resulting modulated waveform is:

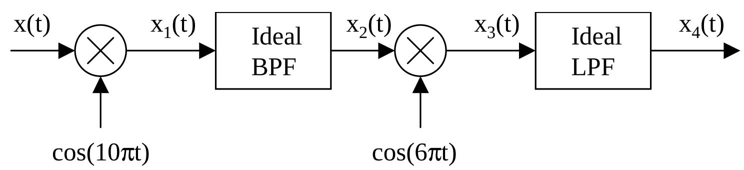
$$x_m(t) = 400\text{sinc}(200t)\cos(2000\pi t)$$

(a) **Sketch** the Fourier Transform $X_m(f)$ of the **modulated** signal.

(b) What **kind** of modulation is this? Be speficic (e.g. DSB-SC, DSB-LC, USSB, or LSSB).

(c) **Design a receiver** for this signal. The input of the receiver is $x_m(t)$ and the output should be identical to $x(t)$. Make sure you accurately sketch the frequency response of any filter(s) you use in your design.

2. Consider the system shown below:



Where the Fourier Transform of the input is:

$$X(f) = 4\Lambda\left(\frac{f}{2}\right)$$

The ideal bandpass filter (BPF) has a passband between 3 and 5 Hz, and the ideal lowpass filter (LPF) has a cutoff at 2 Hz. **Carefully sketch the Fourier Transform of signals $x_1(t)$, $x_2(t)$, $x_3(t)$, and $x_4(t)$. Is the output $x_4(t)$ of this system the same as the input $x(t)$?**

3. Consider a DSB-SC system where the Fourier Transform of the message signal $x(t)$ is:

$$X(f) = 2\Pi\left(\frac{f}{20}\right)$$

and the carrier frequency is $f_c = 30$ Hz.

(a) **Find and sketch the Fourier Transform of the modulated signal $x_m(t)$.**

(b) Assume that the receiver is identical to the coherent receiver studied in class, only the oscillator does not have the exact same frequency as the carrier. More specifically, assume that the receiver's oscillator frequency is $f_o = 29$ Hz. **Find and carefully sketch the Fourier Transform of the output of this receiver. Is the output of the receiver the same as the input to the transmitter?**

4. Consider a DSB-LC system where the message signal is:

$$x(t) \quad = \quad \cos(2\pi t) + \cos(6\pi t)$$

and the modulated signal is:

$$x_m(t) \quad = \quad [A + x(t)]\cos(10\pi t)$$

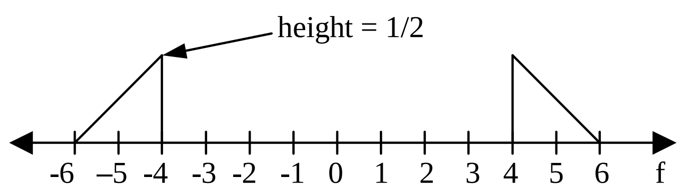
(a) What is the **minimum value of the constant** A for which an envelope detector can be used to recover the message signal $x(t)$ from the modulated signal $x_m(t)$ without distortion?

(b) Using the value for A that you found in part A, **find and sketch the Fourier Transform of the modulated signal** $x_m(t)$.

5. The input to a modulator is a signal $x(t)$ with Fourier Transform

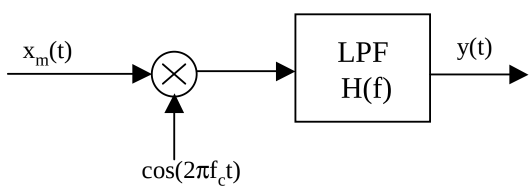
$$X(f) \quad = \quad \Lambda\left(\frac{f}{2}\right)$$

The output $x_m(t)$ of the modulator has Fourier Transform $X_m(f)$ shown below:



(a) What type of modulation is this (DSB-SC, DSB-LC, Upper-SSB, Lower-SSB, FM)?

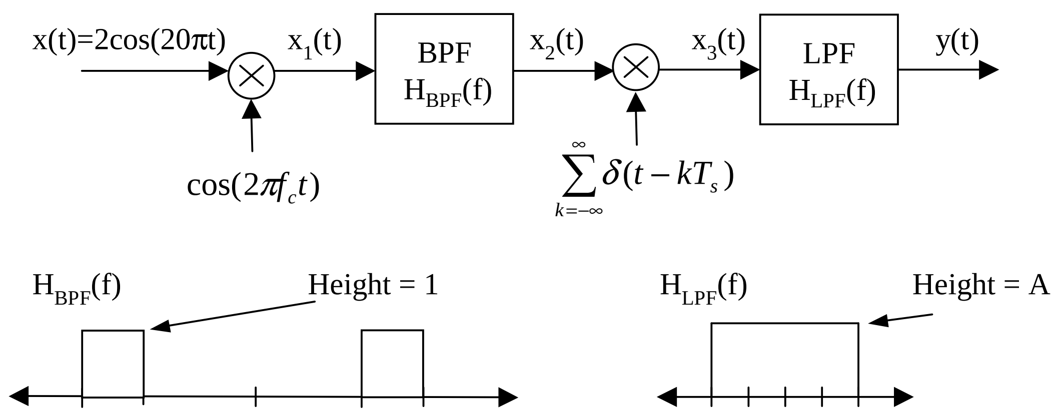
(b) Consider the following coherent receiver:



Determine the value f_c and frequency response of the filter such that $y(t) = x(t)$.

(c) Can $x(t)$ be recovered from $x_m(t)$ with an envelope detector (Yes or No)?

6. Consider the following system:



where the frequency response of the BPF and LPF are as shown.

(a) If $f_s = 1/T_s = 50$ Hz, determine values for the parameters f_c and A such that $y(t) = x(t)$.

(b) For your choice of f_c, what type of modulation is $x_2(t)$?

Chapter 7

Probability

7.1 Prelude: The Monte Hall Puzzler

You have been selected to be a contestant on the TV game show "Let's Make A Deal". The rules of the game are as follows:

- There are three closed doors. Behind one of the doors are the keys to a new car. Behind each of the other two doors is a box of Rice-A-Roni (the San Francisco treat).
- You begin the game by selecting one of the three doors. This door remains closed.
- Once you select a door, the host (Monty Hall) will open up one of the other two doors revealing ... a box of Rice-A-Roni.
- You must choose between keeping the door that you originally selected or switching to the other closed door. The door that you chose is then opened and you are allowed to keep the prize that is behind it (and in the case of the keys, you get ownership of the car).

Answer the following.

1. If you want to maximize your chances of winning the car, should you:
   (a) *Keep* the door you originally selected.
   (b) *Switch* to the other door.
   (c) *Randomly choose* between the two doors.

   You may assume that Monty Hall knows which door has the keys behind it and that he doesn't want you to win the car.
2. What is the probability of winning the car if you stay with your original choice?
3. What is the probability if you switch your choice?
4. What is the probability if you choose between the two closed doors at random?

7.2 Random Signals

Up until now, we have only considered **deterministic** signals, i.e. signals whose parameters are non-random. However, in most practical systems, the signals of interest are unknown. For instance, in a communication system, the message signal is not known and the channel noise is not known. Unknown signals can be modelled, with the aid of probability theory, as **random** signals. The chapter after this one is on random signals. But first, we need to start with a review of probability and random variables.

7.3 Key Terms Related to Probability Theory

7.3.1 Random Experiment

A **random experiment** is an experiment that has an unpredictable outcome, even if repeated under the same conditions. Examples include: A coin toss, a die roll, measuring the exact voltage of a waveform. *Give some other examples of random experiments.*

7.3.2 Outcome

An **outcome** is the result of the random experiment. Example: For a coin toss there are two possible outcomes (heads or tails). Note that the outcomes of a random experiment are mutually exclusive (i.e. you can't toss heads and tails at the same time with a single coin) and that they aren't necessarily numbers (heads and tails are not numbers, are they?).

7.3.3 Sample Space

The **sample space** is the set S of all outcomes. For examples, we write the sample space of a coin toss as:

$$S = \{heads, tails\}.$$

7.3.4 Random Variable

The outcomes of a random experiment are not necessarily represented by a numerical value (i.e. coin toss). A **random variable** (RV) is a *number* that describes the outcome of a random experiment. **Example:** for a coin toss, define a random variable X, where:

$$X = \begin{cases} 0 & \text{if "tails"} \\ 1 & \text{if "heads"} \end{cases}$$

7.4 Cumulative Distribution Function

7.4.1 Definition

The **Cumulative Distribution Function** (CDF) of a random variable X is defined by:

$$F_X(a) = P[X \leq a]$$

where $P[E]$ denotes the probability of event E. In words, the CDF is a function of a that tells us the probability that the random variable X is less than or equal to a.

7.4.2 Example

A "fair" coin toss is represented by the random variable

$$X = \begin{cases} 0 & \text{if "tails"} \\ 1 & \text{if "heads"} \end{cases}$$

Find and sketch the CDF for this RV.

7.4.3 Properties

All valid CDFs have the following properties:

1. $F_X(a)$ is a nondecreasing function of a. That is, as a increases, $F_x(a)$ cannot decrease.

2. $F_X(-\infty) = 0$.

3. $F_X(\infty) = 1$.

4. $P[a < X \leq b] = F_X(b) - F_X(a)$.

7.5 Probability Density Function

7.5.1 Definition

The **probability density function** (pdf) of a random variable X is defined by:

$$f_x(x) = \left.\frac{d}{da}F_X(a)\right|_{a=x}$$

Simply put, the pdf is the derivative of the CDF.

7.5.2 Example

Find the pdf for our example (fair coin toss).

7.5.3 Properties

All valid pdfs have the following properties:

1. The probability that the RV takes on a value within the range $(a, b]$ is found by integrating the pdf over that range:

$$P[a < X \leq b] = \int_a^b f_X(x)dx$$

2. $f_X \geq 0$, i.e. the pdf is nonnegative.

3. The pdf integrates to one, i.e.:

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

7.5.4 Another example

Consider a **uniform** RV that takes on values in the range $(-1, +1)$ with equal probability. The pdf of this RV is:

$$f_X(x) = c\Pi\left(\frac{x}{2}\right)$$

First, find the value of c such that this is a valid pdf.

Next, compute the following probabilities:

1. $P[-1 < X \leq 0]$
2. $P[|X| < 1/4]$
3. $P[X < 1/2]$

7.5.5 Bernoulli Random Variable

The most basic **discrete random variable** is the *Bernouilli* random variable. Consider an *unfair* coin that lands on heads with probability p. Assign a random variable X according to:

$$X = \begin{cases} 0 & \text{if "tails"} \\ 1 & \text{if "heads"} \end{cases}$$

The pdf of this random variable is:

$$f_X(x) = (1-p)\delta(x) + p\delta(x-1)$$

7.5.6 Uniform Random Variable

One of the most basic **continuous random variable** is the *uniform* random variable. A uniform random variable takes on values over the continuous range (a, b) with equal probability. The pdf is as follows:

$$f_X(x) = \frac{1}{b-a}\Pi\left(\frac{x-m}{b-a}\right)$$

where

$$m = \frac{b+a}{2}$$

7.6 Independence

Two random variables, X and Y, are independent if the random experiments that generate the two random variables are independent. In other words, if the outcome of the experiment that produces X has no influence on the outcome of the experiment that produces Y, then X and Y are independent.

7.6.1 Independent and Identically Distributed

Two random variables, X and Y, are said to be "independent and identically distributed" if they are independent and have the same pdf, i.e. $f_Y(\lambda) = f_X(\lambda)$.

7.6.2 Sums of Independent Random Variables

Theorem: Let $Z = X + Y$, where X and Y are independent. Then the pdf of Z is found by convolving the pdfs of X and Y:

$$f_Z(z) = f_X(z) * f_Y(z)$$

7.6.3 Example #1

Let X and Y be i.i.d. random variables that are each uniform over $(-0.5, 0.5)$. Find the pdf of $Z = X + Y$.

7.6.4 Example #2

Let X and Y be two i.i.d. Bernoulli random. Find the pdf of $Z = X + Y$.

7.6.5 A Generalization of the Theorem

Let $X_1, X_2, ..., X_N$ be a set of N independent random variables. Then the pdf of $Y = \sum_{n=1}^{N} X_n$ is

$$f_Y(y) = f_{X_1}(y) * f_{X_2}(y) * ... * f_{X_N}(y)$$

7.6.6 Binomial Random Variables

Let $X_1, X_2, ...X_N$ be a set of i.i.d. Bernoulli random variables, then the pdf of $Y = \sum_{n=1}^{N} X_n$ is

$$\begin{aligned} f_Y(y) &= f_{X_1}(y) * f_{X_2}(y) * ... * f_{X_N}(y) \\ &= [q\delta(y) + p\delta(y-1)] * [q\delta(y) + p\delta(y-1)] * ... * [q\delta(y) + p\delta(y-1)] \\ &= \sum_{k=0}^{N} \begin{pmatrix} N \\ k \end{pmatrix} p^k q^{n-k} \delta(y-k) \end{aligned}$$

where $q = 1 - p$ and $\begin{pmatrix} N \\ k \end{pmatrix}$ is called the **binomial coefficient**, defined by:

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}$$

7.6.7 Example #1

A fair coin is tossed 5 times. Find the probability that 2 or fewer heads were tossed.

7.6.8 Example #2

Data is transmitted over a noisy channel with bit error probability $p = 0.001$. Find the probability that there is more than one bit error in a byte of received data.

7.7 Expectation

The **expected value** or **mean** of a random variable X is defined by:

$$m_X = E[X] \tag{7.1}$$

$$= \int_{-\infty}^{\infty} x f_X(x) dx \tag{7.2}$$

7.7.1 Example #1

Find the mean of an arbitrary uniform random variable.

7.7.2 Example #2

Find the mean of an arbitrary Bernoulli random variable.

7.7.3 Moments

The **n-th moment** of a random variable X is defined by:

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)dx \tag{7.3}$$

Note that the mean is the first moment.

7.7.4 Example #1

Find the second moment of an arbitrary uniform random variable.

7.7.5 Example #2

Find the second moment of an arbitrary Bernoulli random variable.

7.7.6 Variance

The **variance** of a random variable X is defined by:

$$\sigma_X^2 = E[(X - m_X)^2] \tag{7.4}$$

$$= \int_{-\infty}^{\infty} (x - m_X)^2 f_X(x)dx \tag{7.5}$$

7.7.7 Relationship Between Variance and First Two Moments

Variance is a function of the first and second moments:

$$\sigma_X^2 = E[X^2] - m_X^2$$

Proof:

7.7.8 Example #1

Find the variance of an arbitrary uniform random variable.

7.7.9 Example #2

Find the variance of an arbitrary Bernoulli random variable.

7.8 Gaussian RV's

7.8.1 The Central Limit Theorem

Let $X_1, X_2, ..., X_N$ be a set of N i.i.d. continuous random variables (e.g. uniform). Then the pdf of $Y = \sum_{n=1}^{N} X_n$ is

$$f_Y(y) \quad = \quad f_{X_1}(y) * f_{X_2}(y) * ... * f_{X_N}(y)$$

As $N \to \infty$, Y will have a Gaussian distribution:

$$f_Y(y) \quad = \quad \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{ \frac{-(y-m)^2}{2\sigma^2} \right\}$$

where m is the mean of Y and σ^2 is the variance of Y.

This is called the **Cental Limit Theorem** and justifies the use of Gaussian RVs for many engineering applications, e.g. measurement error, receiver noise, interference, etc.

A shorthand notation that is often used is $X \sim \eta\left(m, \sigma^2\right)$. This tells us that X has a **normal** (i.e. Gaussian) distribution with mean m and variance σ^2.

7.8.2 Properties of Gaussian RVs

Gaussian RVs have the following properties:

1. A Gaussian RV is completely described by its mean and variance.

2. The sum of 2 Gaussian RVs is also Gaussian.

3. If X and Y are Gaussian and $E[XY] = E[X]E[Y]$, then X and Y are independent.

4. The pdf of a Gaussian RV is symmetric about its mean, and thus $P[Y \leq m] = P[Y > m] = 1/2$.

7.8.3 Example: Applying the properties of Gaussian RVs

X_1 and X_2 are i.i.d. $\eta(2, 3)$. What is the pdf of $Y = X_1 + X_2$?

What is the median of Y?

If $E[X_1 X_2] = 4$, then are X_1 and X_2 independent?

7.8.4 Computing the CDF of Gaussian RVs

Let X be Gaussian, then its CDF is:

$$F_X(a) = \int_{-\infty}^{a} f_X(x)dx$$

where

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2} dx$$

is called the **error function**. You can compute this in matlab by using the *erf* command.

7.8.5 Area Under the Tail of a Gaussian RV

The complement of the CDF is the "area under the tail".

$$\int_a^\infty f_X(x)dx = 1 - F_X(a)$$

where

$$\text{erfc}(z) = 1 - \text{erfc}(z)$$

is called the **complementary error function**, which can be computed in matlab using *erfc*.

The area under the tail can also be put in terms of the *Q-function*:

$$\begin{aligned} Q(z) &= \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-x^2/2} dx \\ &= \frac{1}{2} \text{erfc}\left(\frac{z}{\sqrt{2}} \right) \end{aligned}$$

In particular,

$$1 - F_X(a) = Q\left(\frac{a-m}{\sigma} \right)$$

7.8.6 Properties of the Q-function

The Q-function has the following properties:

1. $Q(0) = 1/2$.

2. $Q(\infty) = 0$.

3. $Q(-z) = 1 - Q(z)$.

Let $X \sim \eta(m, \sigma^2)$, then

$$F_X(a) = Q\left(\frac{m-a}{\sigma} \right)$$

You can download a matlab script for computing the Q-function from the course web-page, or use the following Q-function table:

z	$Q(z)$	z	$Q(z)$	z	$Q(z)$	z	$Q(z)$
0.0	0.50000	1.0	0.15866	2.0	0.02275	3.0	0.00135
0.1	0.46017	1.1	0.13567	2.1	0.01786	3.1	0.00097
0.2	0.42074	1.2	0.11507	2.2	0.01390	3.2	0.00069
0.3	0.38209	1.3	0.09680	2.3	0.01072	3.3	0.00048
0.4	0.34458	1.4	0.08076	2.4	0.00820	3.4	0.00034
0.5	0.30854	1.5	0.06681	2.5	0.00621	3.5	0.00023
0.6	0.27425	1.6	0.05480	2.6	0.00466	3.6	0.00016
0.7	0.24196	1.7	0.04457	2.7	0.00347	3.7	0.00011
0.8	0.21186	1.8	0.03593	2.8	0.00256	3.8	0.00007
0.9	0.18406	1.9	0.02872	2.9	0.00187	3.9	0.00005

7.8.7 Examples

Let $X \sim \eta(0, 1)$. Compute the following probabilities:

1. $P[X > 2]$

2. $P[X \leq 2]$

3. $P[X > -1]$

4. $P[X \leq -1]$

Let $X \sim \eta(1, 4)$. Compute the following:

1. $P[X > 2]$

2. $P[X \leq 2]$

3. $P[X > -5]$

4. $P[X \leq -5]$

7.9 Exercises

1. For a probability density function (pdf) to be valid, the follow two properties must be satisfied:

i. The pdf is a nonnegative function (i.e. $f_X(x) \geq 0, \forall x$).

ii. The pdf integrates to one.

Determine whether or not each of the following pdfs is valid:

$$
\begin{aligned}
\text{(a):} \quad f_X(x) &= \frac{1}{\pi}\left(\frac{1}{1+x^2}\right) \\
\text{(b):} \quad f_X(x) &= \begin{cases} |x|, & |x| < 1 \\ 0, & \text{otherwise} \end{cases} \\
\text{(c):} \quad f_X(x) &= \begin{cases} \frac{1}{6}(8-x), & 4 \leq x \leq 10 \\ 0, & \text{otherwise} \end{cases} \\
\text{(d):} \quad f_X(x) &= \sum_{k=0}^{\infty} \frac{3}{4}\left(\frac{1}{4}\right)^k \delta(x-k)
\end{aligned}
$$

$$
\text{Hint:} \quad \sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \quad \text{if } |r| < 1
$$

2. A random variable X_1 describes the outcome of a fair die roll.

(a) What is the sample space S of the random experiment used to produce X_1?

(b) Find the CDF of X_1.

(c) Find the pdf of X_1.

(d) Find the probability that $3 < X_1 \leq 5$.

(e) Let X_2 be another random variable that also describes the outcome of a fair die roll. Let the two die rolls that produce X_1 and X_2 be independent. Next, define a random variable $Y = X_1 + X_2$ which describes the outcome when the pair of dice are rolled. Repeat parts 2.a through 2.d, but answer these for Y rather than for X_1.

3. A random variable X is uniform over the region $(-1, 3)$.

(a) Write an expression for the pdf in terms of the $\Pi(\cdot)$ function and sketch the pdf.

(b) Determine $P[X \leq 0]$

(c) Determine $P[X > 1.5]$

(d) Determine $P[-0.5 < X \leq 2]$

4. A random variable X_1 has pdf:

$$f_{X_1}(x) \quad = \quad ce^{-2x}u(x)$$

where $u(x)$ is the unit step function and c is a positive constant.

(a) Determine the value of the constant c required for this to be a valid pdf.

(b) Determine the **median** value of X_1, i.e. the exact value of X_1 for which the random variable is just as likely to be greater than this value as it is to be lower than it.

(c) Let X_2 be a second random variable that is independent from X_1 but has the same pdf. Let $Y = X_1 + X_2$. Find the pdf of Y (hint: you may use your FT table to help with this problem).

5. A random variable X has pdf:

$$f_X(x) \quad = \quad \begin{cases} \frac{3}{32}(-x^2 + 8x - 12), & x_o \leq x \leq 6 \\ 0, & \text{otherwise} \end{cases}$$

(a) Determine the value of x_o that makes $f_X(x)$ a valid pdf.

(b) Find the mean of X, i.e. $E[X]$.

(c) Find the second moment of X, i.e. $E[X^2]$.

(d) Find the variance of X.

(e) Find $P[X \leq 4]$.

6. Let X_1, X_2, and X_3 be independent and identically distributed (i.i.d.) random variables with pdf

$$f_X(x) \quad = \quad \frac{1}{2}\delta(x) + \frac{1}{2}\delta(x-1)$$

and let Y_1 and Y_2 be i.i.d. random variables with pdf

$$f_Y(y) \quad = \quad \Pi(y).$$

The X_i's are independent from the Y_j's. Further define $Z = X_1 + Y_1$. Compute the following quantities (give a numerical value for each):

(a) $P[X_1 \leq 0]$

(b) $P[X_1 + X_2 > 0]$

(c) $P[X_1 + X_2 + X_3 > 1]$

(d) $P[Y_1 \leq 0]$

(e) $P\left[|Y_1 + Y_2| > \frac{1}{2}\right]$

(f) $P[Z \leq 0]$

(g) The mean of Z

(h) The variance of Z

7. A digital communication systems transmits packets of data in frames of 16 bits. The system uses an error correction code that is capable of correcting up to 2 bit errors per frame. In other words, if the frame is received with 2 or fewer errors it can be correctly decoded, but if the frame is received with more than 2 errors it will be incorrectly decoded. If the probability that any one bit is in error is $p = 0.01$, then find the probability that the frame is incorrectly decoded.

8. A random variable X has a Gaussian distribution with mean μ and variance σ^2. Find the probability that:

(a) $|X - \mu| < \sigma$.

(b) $|X - \mu| < 2\sigma$.

(c) $|X - \mu| < 3\sigma$.

9. Let X_1 and X_2 be a pair of independent and identically distributed random variables, each with a Gaussian pdf with a mean of 1 and variance of 4. Let Y be a Gaussian random variable that is created by:

$$Y = X_1 + 3X_2$$

(a) Determine $P[X_1 \leq 0]$.

(b) Determine $P[X_2 \leq 3]$.

(c) Determine $P[Y \leq 4]$.

(d) Determine $P[Y \leq -8]$.

10. X_1 and X_2 are a pair of Gaussian random variables with the following properties:

- X_1 has a mean of zero and variance of 4.
- X_2 has a mean of 4 and a variance of 16.
- $E[X_1 X_2] = 0$.

Compute the following probabilities (give numerical answers to five decimal places):

1. $P\left[X_1 \leq 2\right]$
2. $P\left[|X_1| \leq 4\right]$
3. $P\left[X_2 \leq 2\right]$
4. $P\left[|X_2| \leq 6\right]$
5. $P\left[(X_1 + X_2) \leq 4\right]$
6. $P\left[(X_1 + X_2) \leq 8.47\right]$

11. Error control codes are often used to improve the performance of practical digital communications (e.g. cellular phones) and storage systems (e.g. compact discs, hard drives, and DVDs). Assume that a code can correct t=6 or fewer errors in a block of n coded bits. Also assume that each code bit has an error probability of $p = 0.001$. What is the largest value of n for which the probability that the code word is incorrectly decoded is no more than $P_e = 10^{-6}$? **Hint:** You might want to use MATLAB to help you arrive at your answer.

12. (Problem 2.10 from M.P. Fitz, *Analog Communication Theory*) A random variable X has pdf given by

$$f_X(x) = \begin{cases} 0 & x < 0 \\ K_1 & 0 \le x < 1 \\ K_2 & 1 \le x < 2 \\ 0 & x \ge 2 \end{cases}$$

(a) If the mean of X is 1.2, find K_1 and K_2.

(b) Find the variance of X using the value of K_1 and K_2 computed in (a).

(c) Find the probability that $X \le 1.5$ using the value of K_1 and K_2 computed in (a).

13. (This is Problem 2.11 of Fitz) In communications, the phase shift induced by propagation between transmitter and receiver, Φ is often modeled as a random variable. A common model is to have

$$f_\Phi(\phi) = \begin{cases} \frac{1}{2\pi} & \text{for } -\pi \le \phi \le \pi \\ 0 & \text{elsewhere} \end{cases}$$

This is a uniformly distributed random variable.

(a) Find the CDF of Φ.

(b) Find the mean and variance of Φ.

(c) A particular communication system will work reasonable well if the estimated phase $\hat{\Phi}$ is within 30^o of the true value of Φ. If you implement a receiver with $\hat{\Phi} = 0$, what is the probability that the communication system will work?

(d) Assume that if you move the receiver's antenna, you randomly change the propagation delay and obtain an independent phase shift. What is the probability that the system will work in at least one out of two antenna locations?

(e) How many antenna locations would you have to try to ensure a 90% chance of getting the system to work?

14. A biometrics system is used to determine whether or not a certain subject is "Nicholas". The system scans the fingerprint of the subject and compares the fingerprint against a stored template for Nicholas. Based on how close the scanned fingerprint is to the template, it generates a matching score X, which is a random variable. The system declares that the individual is Nicholas if $X > 1$, otherwise it declares that the individual is not Nicholas. If the subject is Nicholas, then X is Gaussian with mean $m = 3$ and variance $\sigma^2 = 4$. Otherwise, if the subject is somebody else, then X is uniformly distributed over the range $-b < X < b$.

(a) What is the probability that the system makes a mistake when it scans Nicholas's fingerprint (i.e. it thinks that the subject is not Nicholas, even though it actually is)?

(b) Suppose that the system makes a mistake 10% of the time when the subject is not Nicholas (i.e. it declares that the subject is Nicholas, even though it is not). What is the numerical value of b in this case?

Chapter 8

Random Processes

8.1 Random Variables versus Random Processes

Recall our definition of a random variable: "A **random variable** (RV) is a *number* that describes the outcome of a random experiment".

The definition of a random process is very similar: "A **random process** (RP) is a *function of time* (or *signal*) that describes the outcome of a random experiment.

For instance, let the random experiment be a coin toss and then define a random process as:

$$x(t) = \begin{cases} \sin(t) & \text{if "tails"} \\ \cos(t) & \text{if "heads"} \end{cases}$$

Note that a random process evaluated at a particular instance in time is a random variable, for instance let's evaluate (or sample) this random process at time $t = 0$, then we get:

$$x(0) = \begin{cases} \sin(0) = 0 & \text{if "tails"} \\ \cos(0) = 1 & \text{if "heads"} \end{cases}$$

which is a random variable.

8.2 Describing a Random Process

Recall that a random variable can be described in terms of its mean and variance. Since a random variable is a number, the mean and variance is also a number. Likewise, we can describe a random process in terms of its mean and variance, only now these two become functions. More specifically, we can use the *mean* and the *autocorrelation function.*

8.2.1 Mean

The mean of a random process is defined as $m_x(t) = E[x(t)]$. In general, the mean may be a function of time. However, for some processes it turns out that the mean is a constant.

In general it is difficult to compute the mean because the pdf itself could change over time. However, many random processes can be expressed as a function of a random variable. In this case, it is actually quite easy to find the mean of the random process.

Example: Consider the random process $x(t) = \cos(2\pi t + \theta)$ where θ is a random variable which is uniformly distributed over the range $(0, 2\pi)$. Compute the mean of $x(t)$.

8.2.2 Autocorrelation Function

Since the mean of a RP can be a function of time, we would expect the same to be true for the variance of the RP. Thus, we *could* define the variance of a RP as being $\sigma^2(t) = E[(x(t) - m_X(t))^2]$. However, for random processes it is more common to instead use a function called the autocorrelation function (ACF) which is defined as follows:

$$R_x(t_1, t_2) = E[x(t_1)x(t_2)]$$

Note that the ACF is actually a function of two times, and in that sense is actually more general than the variance (which would only be a function of just one time). Because it is a function of two times, the ACF tells us how the RP changes over time.

Example: Determine the ACF for $x(t) = \cos(2\pi t + \theta)$, where $\theta \sim U(0, 2\pi)$.

8.2.3 Stationarity

A random process is said to be **wide sense stationary** (WSS) if:

1. The mean is a constant

$$m_x(t) = m_x$$

2. The ACF is a function of time difference only

$$R_x(t_1, t_2) = R_x(\tau)$$

where $\tau = t_2 - t_1$

Example: Is $x(t) = \cos(2\pi t + \theta)$ WSS when $\theta \sim U(0, 2\pi)$?

8.2.4 Power Spectral Density

So what does knowing that a RP is WSS "buy" us? Well, actually quite a bit, because it allows us to find a frequency-domain representation of the RP. In particular, if a RP is WSS then its **power spectral density** (PSD) is the Fourier Transform of its ACF, i.e.

$$S_x(f) = \mathcal{F}\{R_x(\tau)\} \tag{8.1}$$

This relationship is called the *Wiener-Khintchine theorem*.

Example: Determine the PSD for $x(t) = \cos(2\pi t + \theta)$, where $\theta \sim U(0, 2\pi)$.

8.2.5 Parseval's Theorem

As the name implies, the PSD tells us how much power there is per unit bandwidth. In fact, PSD has units of Watts per Hertz (or W/Hz). Thus, if we want to know how much power resides within a certain bandwidth, we can simply integrate the PSD over that bandwidth. Or, if we would like to know the total power of the signal, then simply integrate over all frequency:

$$P = \int_{-\infty}^{\infty} S_x(f) df \tag{8.2}$$

Thus, the PSD gives us another way to compute the power of a signal.

Example: Compute the power of $x(t) = \cos(2\pi t + \theta)$, where $\theta \sim U(0, 2\pi)$.

8.3 LTI Systems with Random Inputs

We know that when a LTI system with impulse response $h(t)$ is stimulated with a deterministic signal $x(t)$, then we can find the output $y(t)$ by simply performing the convolution:

$$y(t) \quad = \quad x(t) * h(t) \tag{8.3}$$

Alternatively, we could work in the frequency domain:

$$Y(f) \quad = \quad X(f)H(f) \tag{8.4}$$

But what if $x(t)$ is a random process? How can we find its output?

Well if $x(t)$ is random, then so will $y(t)$ and therefore it is impossible to tell exactly what the output will be. However, if $x(t)$ is WSS then we can determine its PSD. Thus, it makes more sense to think in terms of PSD. In particular, the PSD of the output is related to the PSD of the input by:

$$S_y(f) = S_x(f)|H(f)|^2 \tag{8.5}$$

Example: The input to a LTI filter is a WSS random process with ACF:

$$R_x(\tau) \quad = \quad 10\text{sinc}^2(1000\tau)$$

The frequency response of the filter is $H(f) = 10\Pi(f/1000)$. Determine the PSD of the signal at the filter output and the corresponding total power.

8.4 White Gaussian Noise

One particular type of random process is so common, that it should be mentioned here. A *Gaussian* RP is simply a RP for which every sample is a Gaussian RV. A process is said to be *white* if it contains all frequency components with equal power, or in other words if its PSD is a constant (i.e. flat). A process $n(t)$ is said to be *white Gaussian* if:

1. Samples of $n(t)$ are Gaussian

2. $n(t)$ is WSS

3. $m_n(t) = 0$

4. $S_n(f) = K$, a constant

5. $R_n(\tau) = \mathcal{F}^{-1}\{K\} = ?$

Noise is often modelled as a white Gaussian process, in which case it is called *white Gaussian noise*. In this case, then the constant $K = N_o/2$, where the quantity $N_o/2$ is called the *two-sided noise spectral density* and N_o is called the *one-sided noise spectral density*.

One other thing: If the input to a LTI system is a Gaussian RP, then the output will also be a Gaussian RP (although it won't necessarily be white ... it could be *colored* noise).

Example: The input to a filter is a white Gaussian noise process $n(t)$ with two-sided noise spectral density $N_o/2 = 10^{-3}$ W/Hz. The frequency response of the filter is $H(f) = 10\Pi(f/1000)$. Determine the PSD of the colored Gaussian noise at the filter output and the corresponding total power.

8.5 Signal-to-Noise Ratio

Suppose that the input to a LTI system is:

$$x(t) = s(t) + n(t).$$

Where $s(t)$ is the desired signal and $n(t)$ is noise.

The output of the system is:

$$\begin{aligned} y(t) &= [s(t) + n(t)] * h(t) \\ &= s(t) * h(t) + n(t) * h(t) \\ &= s'(t) + n'(t), \end{aligned}$$

where $s'(t)$ is the portion of the output due to the signal and $n'(t)$ is the portion of the output due to noise.

The (output) signal-to-noise ratio (SNR) is then:

$$SNR = \frac{P_{s'}}{P_{n'}}, \tag{8.6}$$

where $P_{s'}$ is the power of $s'(t)$ and $P_{n'}$ is the power of $n'(t)$.

Oftentimes, the SNR is expressed in dB,

$$\begin{aligned} SNR &= 10 \log_{10} \frac{P_{s'}}{P_{n'}} && (8.7) \\ &= 10 \log_{10} P_{s'} - 10 \log_{10} P_{n'} && (8.8) \end{aligned}$$

Example The input to a LTI system is $x(t) = s(t) + n(t)$ where the $s(t)$ is a WSS process with ACF:

$$R_s(\tau) = 10\text{sinc}^2(1000\tau),$$

$n(t)$ is white Gaussian noise with two-sided noise spectral density $N_o/2 = 10^{-3}$ W/Hz, and the frequency response of the filter is $H(f) = 10\Pi(f/1000)$. Determine the SNR at the output of the filter.

8.6 Exercises

1. Given a function $x(t) = 2\sin(2\pi(1000)t + \Theta)$ where Θ is uniformly distributed between $-\pi$ and π:

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & \text{for } -\pi < \theta \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the mean of $x(t)$

(b) Find the autocorrelation of $x(t)$

(c) Is $x(t)$ Wide Sense Stationary (WSS)? If so, find the Power Spectral Density (PSD) of $x(t)$.

2. Additive white Gaussian noise $n(t)$ with two-sided spectral density $N_0/2 = 0.1$ W/Hz is input into a linear time invariant filter with frequency response $H(f) = 2\Pi(f/20)$. The output is $n'(t) = n(t) * h(t)$.

(a) Determine the autocorrlation $R_n(\tau)$ of $n(t)$.

(b) Determine $m_{n'}(t) = E[n'(t)]$.

(c) Determine the power spectral density at the filter output $S_{n'}(f)$.

(d) Determine $E\left[(n'(t))^2\right]$.

(e) Give the pdf of the output sampled at time t_1.

(f) Give an expression for $P[n'(t_1) > 3]$.

3. The input to a LTI filter is $x(t) = s(t) + n(t)$ where $n(t)$ is additive white Gaussian noise with two-sided spectral density $N_0/2 = 10^{-3}$ W/Hz, and $s(t)$ is a WSS process with autocorrelation:

$$R_s(\tau) = 6000\text{sinc}(3000\tau).$$

The filter has frequency response:

$$H(f) = \begin{cases} 5 & |f| < 1000 \\ 2 & 1000 \leq |f| < 2000 \\ 0 & \text{elsewhere} \end{cases}$$

Determine the SNR at the output of the filter. Express your answer in dB.

Appendix A

Useful Functions and Tables

A.1 Function Definitions

A.1.1 Unit-Step Function

$$u(\lambda) = \begin{cases} 0 & \lambda < 0 \\ 1 & \lambda > 0 \end{cases}$$

A.1.2 Rectangular Pulse of Width τ

$$\Pi\left(\frac{\lambda}{\tau}\right) = \begin{cases} 1 & |\lambda| < \tau/2 \\ 0 & |\lambda| > \tau/2 \end{cases}$$

A.1.3 Triangular Pulse of Width 2τ

$$\Lambda\left(\frac{\lambda}{\tau}\right) = \begin{cases} 1 - \frac{|\lambda|}{\tau} & |\lambda| < \tau \\ 0 & |\lambda| > \tau \end{cases}$$

A.1.4 Sinc Function

$$\begin{aligned} \operatorname{sinc}(\lambda) &= \begin{cases} 1 & \text{for } \lambda = 0 \\ \frac{\sin(\pi\lambda)}{\pi\lambda} & \text{for } \lambda \neq 0 \end{cases} \\ &= \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} e^{\pm j2\pi s} ds \end{aligned}$$

A.1.5 Sampling Function (Sine-over-Argument)

$$\operatorname{Sa}(\lambda) = \operatorname{sinc}\left(\frac{\lambda}{\pi}\right) = \begin{cases} 1 & \text{for } \lambda = 0 \\ \frac{\sin(\lambda)}{\lambda} & \text{for } \lambda \neq 0 \end{cases}$$

A.1.6 Dirac Delta Function

Definition I: The delta function is the derivative of the unit-step function:

$$\delta(\lambda) \;=\; \frac{d}{d\lambda}u(\lambda)$$

Definition II: Must satisfy both of the following conditions:

1. $\delta(\lambda) = 0$ for $\lambda \neq 0$.
2. $\int_{-\infty}^{\infty} \delta(\lambda) d\lambda = 1$.

Note that these conditions are satisfied by $\frac{1}{\tau}\Pi\left(\frac{\lambda}{\tau}\right)$ as $\tau \to 0$.

Properties of the delta function:

Even function: $\delta(\lambda) = \delta(-\lambda)$.

Integral: For any $\epsilon > 0$, $\int_{-\epsilon}^{\epsilon} \delta(\lambda) d\lambda = 1$.

Multiplication with another function: $g(\lambda)\delta(\lambda) = g(0)\delta(\lambda)$.

Sifting property: $\int_{-\infty}^{\infty} g(\lambda)\delta(\lambda) d\lambda = g(0) \int_{-\infty}^{\infty} \delta(\lambda) dt = g(0)$.

Corresponding properties of shifted delta function:

1. For any $\epsilon > 0$, $\int_{\lambda_o - \epsilon}^{\lambda_o + \epsilon} \delta(\lambda - \lambda_o) d\lambda = 1$
2. $g(\lambda)\delta(\lambda - \lambda_o) = g(\lambda_0)\delta(\lambda - \lambda_o)$
3. $\int_{-\infty}^{\infty} g(\lambda)\delta(\lambda - \lambda_o) d\lambda = g(\lambda_o)$

Multiplication of two delta functions:

$$\delta(\lambda - \lambda_1)\delta(\lambda - \lambda_2) \;=\; \begin{cases} \delta(\lambda - \lambda_1) & \text{if } \lambda_1 = \lambda_2 \\ 0 & \text{if } i \neq j \end{cases}$$

A.2 F.T. Definitions

$$
\begin{aligned}
V(f) &= \int_{-\infty}^{\infty} v(t)e^{-j2\pi ft}dt && \text{Definition of F.T.} \\
v(t) &= \int_{-\infty}^{\infty} V(f)e^{j2\pi ft}df && \text{Definition of Inverse F.T.}
\end{aligned}
$$

A.3 F.T. Pairs

$$
\begin{aligned}
\Pi\left(\frac{t}{\tau}\right) &\Leftrightarrow \tau\mathrm{sinc}\,(f\tau) \\
e^{j\omega_o t} &\Leftrightarrow \delta(f-f_o) \\
\delta(t-t_o) &\Leftrightarrow e^{-j2\pi f t_o} \\
\sum_{k=-\infty}^{\infty}\delta(t-kT_o) &\Leftrightarrow f_o\sum_{k=-\infty}^{\infty}\delta(f-kf_o) \\
\delta(t) &\Leftrightarrow 1 \\
K &\Leftrightarrow K\delta(f) \\
\mathrm{sinc}(2Wt) &\Leftrightarrow \frac{1}{2W}\Pi\left(\frac{f}{2W}\right) \\
\cos(\omega_o t) &\Leftrightarrow \frac{1}{2}\delta(f-f_o)+\frac{1}{2}\delta(f+f_o) \\
\sin(\omega_o t) &\Leftrightarrow \frac{1}{2j}\delta(f-f_o)-\frac{1}{2j}\delta(f+f_o) \\
u(t) &\Leftrightarrow \frac{1}{j2\pi f}+\frac{1}{2}\delta(f) \\
\Lambda\left(\frac{t}{\tau}\right) &\Leftrightarrow \tau\mathrm{sinc}^2(f\tau) \\
\sum_{k=-\infty}^{\infty}\Pi\left(\frac{t-kT_o}{\tau}\right) &\Leftrightarrow \tau f_o\sum_{k=-\infty}^{\infty}\mathrm{sinc}(kf_o\tau)\delta(f-kf_o) \\
\mathrm{sinc}^2(2Wt) &\Leftrightarrow \frac{1}{2W}\Lambda\left(\frac{f}{2W}\right) \\
e^{-at}u(t) &\Leftrightarrow \frac{1}{a+j2\pi f} \\
te^{-at}u(t) &\Leftrightarrow \frac{1}{(a+j2\pi f)^2}
\end{aligned}
$$

A.4 F.T. Properties

$ax(t) + by(t)$	$\Leftrightarrow$	$aX(f) + bY(f)$	Superposition (or Linearity)
$X(t)$	$\Leftrightarrow$	$x(-f)$	Duality
$\sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t}$	$\Leftrightarrow$	$\sum_{n=-\infty}^{\infty} c_n \delta(f - nf_o)$	F.T. of periodic function
$\frac{d^n}{dt^n} x(t)$	$\Leftrightarrow$	$(j2\pi f)^n X(f)$	Differentiation in time
$x(t - t_o)$	$\Leftrightarrow$	$e^{-j2\pi f t_o} X(f),$	Time shifting
$x(at)$	$\Leftrightarrow$	$\frac{1}{\|a\|} X\left(\frac{f}{a}\right)$	Time scaling
$\int_{-\infty}^{t} x(\lambda) d\lambda$	$\Leftrightarrow$	$\frac{1}{j2\pi f} X(f) + \frac{X(0)}{2} \delta(f)$	Integration
$x(t) * y(t)$	$\Leftrightarrow$	$X(f)Y(f)$	Convolution
$x(t)y(t)$	$\Leftrightarrow$	$X(f) * Y(f)$	Multiplication
$e^{j\omega_o t} x(t)$	$\Leftrightarrow$	$X(f - f_o)$	Frequency shifting
$t^n x(t)$	$\Leftrightarrow$	$(-j2\pi)^{-n} \frac{d^n}{df^n} X(f)$	Differentiation in frequency
$x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$	$\Leftrightarrow$	$f_s \sum_{k=-\infty}^{\infty} X(f - kf_s)$	Impulse sampling
$x(t) \cos(\omega_c t)$	$\Leftrightarrow$	$\frac{1}{2} X(f + f_c) + \frac{1}{2} X(f - f_c)$	Modulation