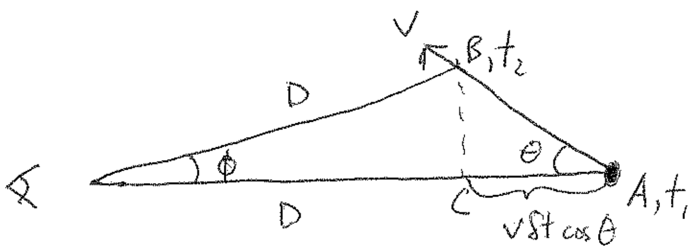
HW 1, PHYS/ASTR 704, Spring 2015 (McWilliams)

1) Carroll 1.4

Derive an expression for $v_{app}(v,\theta)$, where v is the speed of gas expelled by a quasar, θ is the angle of the gas jet relative to the line of sight, and v_{app} is the speed projected onto the line of sight. Show that v_{app} can be > 1.



$t_2 - t_1 \equiv \delta t$

$AB = v\,\delta t$

$AC = v\,\delta t\cos\theta$

$BC = v\,\delta t\sin\theta$

$$t_1' = t_1 + \frac{D + v\,\delta t\cos\theta}{c} \qquad t_2' = t_2 + \frac{D}{c}$$

$$\delta t' = t_2' - t_1' = t_2 - t_1 - \frac{v\,\delta t\cos\theta}{c} = \delta t - \frac{v\,\delta t\cos\theta}{c} = \delta t(1-\beta\cos\theta)$$

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where $\beta \equiv \frac{v}{c}$

$$BC = D\sin\phi \sim \phi D = v\,\delta t\sin\theta = \frac{v\,\delta t'\sin\theta}{1-\beta\cos\theta}$$

$$v_{app} = \frac{\phi D}{\delta t'} = \frac{v\sin\theta}{1-\beta\cos\theta} \quad 10 \qquad \beta_{app} = \frac{v_{app}}{c} = \frac{\beta\sin\theta}{1-\beta\cos\theta} \quad 4$$

$$\frac{\partial\beta_{app}}{\partial\theta} = \frac{\beta\cos\theta}{1-\beta\cos\theta} - \frac{(\beta\sin\theta)^2}{(1-\beta\cos\theta)^2} = 0 \Rightarrow \sin\theta_{max} = \sqrt{1-\cos^2\theta_{max}} = \sqrt{1-\beta^2}$$

$$\therefore \beta_{app,max} = \frac{\beta\sin\theta_{max}}{1-\beta\cos\theta_{max}} = \frac{\beta\sqrt{1-\beta^2}}{1-\beta^2} = \frac{\beta}{\sqrt{1-\beta^2}} = \beta\gamma,$$ where γ is the Lorentz factor

$\therefore$ if $\gamma = \frac{1}{\sqrt{1-\beta^2}} > 1$, v_{app} can be $> c$. 1

2.) Carroll 1.5

A muon has $m = 0.106$ GeV and lifetime $\tau = 2.19 \times 10^{-6}$ s. If such a muon was moving in a circle with a 1 km diameter fast enough to have a total energy of 1000 GeV, how long would it appear to live, and how many rads would it travel.

$$\tau' = \gamma \tau \qquad E = \sqrt{m^2 + \vec{p}^{\,2}} = \sqrt{m^2 + (\gamma m v)^2} = \sqrt{\cancel{m^2} + (\gamma m)^2 - \cancel{\left(\frac{\gamma m}{\gamma}\right)^2}}$$

$$\therefore \gamma = \frac{E}{m} = \frac{1000 \text{ GeV}}{0.106 \text{ GeV}} = 9434 \quad 8$$

$$\therefore \tau' = 9434\,(2.19 \times 10^{-6}\,\text{s}) \simeq 0.02 \text{ s} \quad 7$$

$$\theta = \frac{v\tau'}{r} = \sqrt{1 - \frac{1}{\gamma^2}}\,\frac{\tau' c}{r} \simeq 1.2 \times 10^4 \text{ rad} \quad 7$$

3.) Carroll 1.10

Show how the Maxwell tensor transforms under a) a rotation about the y-axis and b) a boost in the z-direction.

a) $F'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{k}} \frac{\partial x'^{\nu}}{\partial x^{\ell}} F^{\mu\nu} = \Lambda^{\mu}{}_{k} \Lambda^{\nu}{}_{\ell} F^{\mu\nu}$, or in matrix form, $F' = \Lambda F \Lambda^T$

$$\Lambda^{\mu}{}_{k} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix} \qquad F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

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$$\therefore F'^{\mu\nu} = \begin{pmatrix} 0 & -E_x\cos\theta - E_y\sin\theta & -E_y & E_x\sin\theta - E_z\cos\theta \\ E_x\cos\theta + E_y\sin\theta & 0 & B_z\cos\theta - B_x\sin\theta & B_y \\ E_y & -B_z\cos\theta + B_x\sin\theta & 0 & B_z\sin\theta + B_x\cos\theta \\ -E_x\sin\theta + E_z\cos\theta & B_y & -B_z\sin\theta - B_x\cos\theta & 0 \end{pmatrix}$$

3 cont.) $\therefore E^\mu \to E'^\mu = (E_x \cos\theta + E_z \sin\theta,\ E_y,\ E_z \cos\theta - E_x \sin\theta)$

5 $\quad B^\mu \to B'^\mu = (B_z \sin\theta + B_x \cos\theta,\ B_y,\ B_z \cos\theta - B_x \sin\theta)$

b) As in a), except with

$$\Lambda^\mu{}_\kappa = \begin{pmatrix} \cosh\phi & 0 & 0 & -\sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\phi & 0 & 0 & \cosh\phi \end{pmatrix} = \Lambda^\nu{}_\lambda$$

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$$\therefore F'^{\mu\nu} = \begin{pmatrix} 0 & -E_x\cosh\phi - B_y\sinh\phi & -E_y\cosh\phi + B_x\sinh\phi & -E_z \\ E_x\cosh\phi + B_y\sinh\phi & 0 & B_z & -E_x\sinh\phi - B_y\cosh\phi \\ E_y\cosh\phi - B_x\sinh\phi & -B_z & 0 & -E_y\sinh\phi + B_x\cosh\phi \\ E_z & E_x\sinh\phi + B_y\cosh\phi & E_y\sinh\phi - B_x\cosh\phi & 0 \end{pmatrix}$$

$\therefore E^\mu \to E'^\mu = (E_x\cosh\phi + B_y\sinh\phi,\ E_y\cosh\phi - B_x\sinh\phi,\ E_z)$

5 $\quad B^\mu \to B'^\mu = (-E_y\sinh\phi + B_x\cosh\phi,\ E_x\sinh\phi + B_y\cosh\phi,\ B_z)$

NB: Unlike lengths, E^μ and B^μ are only altered in directions that are perpendicular to the motion.

4.) Prove that $\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}(g_{\delta\gamma,\beta} + g_{\beta\delta,\gamma} - g_{\beta\gamma,\delta})$, using

① $g_{\alpha\beta,\gamma} = \Gamma^\sigma_{\alpha\gamma} g_{\sigma\beta} + \Gamma^\sigma_{\beta\gamma} g_{\sigma\alpha}$ and ② $g_{\alpha\beta} g^{\beta\gamma} = \delta_\alpha^\gamma$.

Using ①, $\frac{1}{2} g^{\alpha\delta}(g_{\delta\gamma,\beta} + g_{\beta\delta,\gamma} - g_{\beta\gamma,\delta}) = \frac{1}{2} g^{\alpha\delta}(\Gamma^\sigma_{\delta\beta} g_{\sigma\gamma}$ 5

$+ \Gamma^\sigma_{\gamma\beta} g_{\sigma\delta} + \Gamma^\sigma_{\beta\gamma} g_{\sigma\delta} + \Gamma^\sigma_{\delta\gamma} g_{\sigma\beta} - \Gamma^\sigma_{\beta\delta} g_{\sigma\gamma} - \Gamma^\sigma_{\gamma\delta} g_{\sigma\beta})$

change dummies

$= \frac{1}{2} g^{\alpha\delta}(\Gamma^\alpha_{\delta\beta} g_{\alpha\gamma} + \Gamma^\alpha_{\gamma\beta} g_{\alpha\delta} + \Gamma^\alpha_{\beta\gamma} g_{\alpha\delta} + \Gamma^\alpha_{\delta\gamma} g_{\alpha\beta} - \Gamma^\alpha_{\beta\delta} g_{\alpha\gamma} - \Gamma^\alpha_{\gamma\delta} g_{\alpha\beta})$ 5

using ②

$= \frac{1}{2}(\delta^\delta_\gamma \Gamma^\alpha_{\delta\beta} + \Gamma^\alpha_{\gamma\beta} + \Gamma^\alpha_{\beta\gamma} + \Gamma^\alpha_{\delta\gamma}\delta^\delta_\beta - \Gamma^\alpha_{\beta\delta}\delta^\delta_\gamma - \Gamma^\alpha_{\gamma\delta}\delta^\delta_\beta)$ 5

$= \frac{1}{2}(\Gamma^\alpha_{\gamma\beta} + \Gamma^\alpha_{\beta\gamma}) = \Gamma^\alpha_{\beta\gamma}$ QED

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5.) Prove that $f^\mu = -q U^\lambda F_\lambda{}^\mu \to \vec{f} = q(\vec{E} + \vec{v} \times \vec{B})$ for $v \ll c$.

$U^\lambda = \gamma(1, \vec{v}) \to (1, \vec{v})$ for $v \ll c$ 5

$$F_\lambda{}^\mu = F^{\nu\mu} g_{\lambda\mu} = F^{\nu\mu} \eta_{\lambda\mu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$ 10

$$\therefore f^\mu = -q U^\lambda F_\lambda{}^\mu = -q(-\vec{v}\cdot\vec{E}, -E_x - v_y B_z + v_z B_y, -E_y + v_x B_z - v_z B_x, -E_z - v_x B_y + v_y B_x)$$ 5

$$\overset{v \ll c}{=} -q(0, -\vec{E} - \vec{v} \times \vec{B} = q(\vec{E} + \vec{v} \times \vec{B})$$

QED