HW2, ASTR 704, Spring 2015 (McWilliams)

1) Derive Einstein's equation, $G^{\mu\nu} \approx T^{\mu\nu}$, with a clear written explanation of the steps, particularly why $G^{\mu\nu}$ has the form that it does (i.e. why the Riemann tensor doesn't appear directly).

NB: The 25 points for this one will be distributed subjectively/generously.

We need an equation to relate curvature of spacetime to energy-momentum.

i.e. $G^{\mu\nu}(g^{\mu\nu}, R^{\mu\nu\sigma\rho}, R^{\mu\nu}, R, \text{etc.}) \propto T^{\mu\nu}$, where we know $T^{\mu\nu}{}_{;\alpha} = 0$.

Clearly $R^{\mu\nu} \propto T^{\mu\nu}$ works as far as indices, but $R^{\mu\nu}{}_{;\alpha} \neq 0$.

We therefore need some contraction of the Riemann tensor whose covariant derivative vanishes.

We can start with the Bianchi identity:

$$R_{\alpha\beta\gamma\delta;\epsilon} + R_{\alpha\beta\epsilon\gamma;\delta} + R_{\alpha\beta\delta\epsilon;\gamma} = 0$$

We can operate on this LHS with $g^{\alpha\mu}$, and pull $g^{\alpha\mu}$ inside the covariant derivative, since $g^{\alpha\mu}{}_{;\lambda} = 0$:

$$(g^{\alpha\mu} R_{\alpha\beta\mu\nu})_{;\lambda} + (g^{\alpha\mu} R_{\alpha\beta\lambda\mu})_{;\nu} + (g^{\alpha\mu} R_{\alpha\beta\nu\lambda})_{;\mu} = 0$$

Since $g^{\alpha\mu}$ raises the α indices:

$$R^{\mu}{}_{\beta\mu\nu;\lambda} + R^{\mu}{}_{\beta\lambda\mu;\nu} + R^{\mu}{}_{\beta\nu\lambda;\mu} = 0$$

1 cont.) We can use an (anti-) symmetry of Riemann to rewrite:

$$R^{\mu}{}_{\beta\mu\nu;\lambda} - R^{\mu}{}_{\beta\mu\lambda;\nu} + R^{\mu}{}_{\beta\nu\lambda;\mu} = 0$$

The first two terms can have μ contracted, and we can operate on the new LHS with the metric $g^{\beta\nu}$:

$$g^{\beta\nu}\left[R_{\beta\nu;\lambda} - R_{\beta\lambda;\nu} + R^{\mu}{}_{\beta\nu\lambda;\mu}\right] = 0$$

Raising indices with the metric:

$$R^{\nu}{}_{\nu;\lambda} - R^{\nu}{}_{\lambda;\nu} - R^{\mu\nu}{}_{\nu\lambda;\mu}$$

$$= R_{;\lambda} - R^{\mu}{}_{\lambda;\mu} - R^{\mu}{}_{\lambda;\mu} = 0$$

$\therefore$ $2R^{\mu}{}_{\lambda;\mu} - R_{;\lambda} = 0$. Using the metric to swap indices:

$$2R^{\mu}{}_{\lambda;\mu} - g^{\mu}{}_{\lambda}R_{;\mu} = \left(2R^{\mu}{}_{\lambda} - g^{\mu}{}_{\lambda}R\right)_{;\mu} = 0$$

Dividing through by 2 and raising λ with the metric:

$$g^{\lambda\nu}\left(R^{\mu}{}_{\lambda} - \tfrac{1}{2}g^{\mu}{}_{\lambda}R\right)_{;\mu} = \left(R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R\right)_{;\mu} = 0$$

$\therefore$ The combination $R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$ has a vanishing covariant derivative and matches indices with $T^{\mu\nu}$.

$\therefore$ $G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R \propto T^{\mu\nu}$ is the simplest equation that satisfies the requirements of linking curvature to matter covariantly.

2.) Calculate all of the nonzero Christoffel symbols of the Schwarzschild spacetime explicitly.

To save myself time and ink, the following page, based on Moore, "A General Relativity Notebook", does a nice job:

physicspages.com/2013/12/25/christoffel-symbols-for-schwarzschild-metric

Grading will be ~ 2 points per symbol, with the rest being awarded for the right general approach and displaying awareness of symmetries.

3.) Fill in the following with g's and ∂x's as needed:

$$T^{\alpha\beta}{}_{\gamma} = \underline{g^{\delta\beta}}\; T^{\alpha}{}_{\delta\gamma} \qquad 6$$

$$T_{\alpha}{}^{\beta\gamma} = \underline{g_{\alpha\delta}}\; T^{\delta\beta\gamma} \qquad 6$$

$$T^{\alpha\beta}{}_{\beta} = \underline{\text{(nothing)}}\; T^{\alpha} \qquad 6$$

$$T^{\alpha}{}_{\beta} = \underline{\frac{\partial x^{\alpha}}{\partial x'^{\gamma}} \frac{\partial x'^{\delta}}{\partial x^{\beta}}}\; T'^{\gamma}{}_{\delta} \qquad 7$$

4.) If a space ship orbits a Schwarzschild black hole at $r = 8M$, what is the orbital period measured by a) an astronaut on the ship, and b) a distant observer?

a) $u_\alpha u^\alpha = -1 = -\left(1 - \frac{2M}{r}\right)(u^t)^2 + r^2 (u^\phi)^2$

$$u^\phi = \Omega u^t = \sqrt{\frac{M}{r^3}}\; u^t$$

$$\therefore u_\alpha u^\alpha = -1 = \left(1 - \frac{3M}{r}\right)(u^t)^2 \qquad 6$$

$$P_{ship} = \frac{2\pi}{u^\phi} = \frac{2\pi}{\Omega u^t} = \frac{P_{obs}}{u^t} \overset{\text{from b)}}{=} 142M\sqrt{1 - \frac{3M}{r}} = \sqrt{\frac{5}{8}}\, 142M = 112M$$

$$= 5.5 \times 10^{-4} \left(\frac{M}{M_\odot}\right) s \qquad 6$$

$$\cancel{\overset{\text{from b)}}{=} \frac{142M}{\sqrt{1-\frac{3M}{r}}} = \frac{142M}{\sqrt{1-\frac{3}{8}}} = \frac{142M}{\sqrt{\frac{5}{8}}}}$$

b) $P_{obs} = \frac{2\pi}{\Omega} = 2\pi\, 8^{3/2} M = 142M = 7 \times 10^{-4} \left(\frac{M}{M_\odot}\right) s \qquad 8$