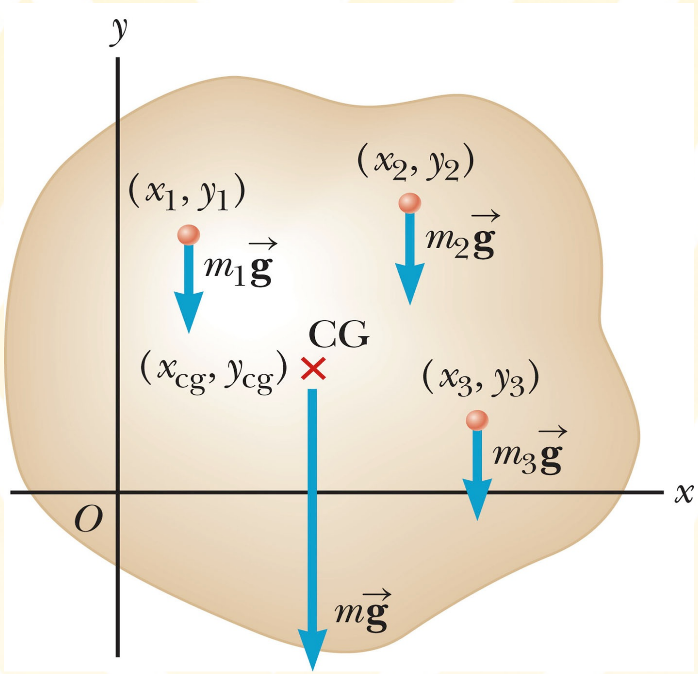
Center of gravity

We divide the rigid body into mass segments and calculate the torque acting on every individual segment. Then, we look for a point, CG, where we can apply the force of gravity to get the same torque.



x - component: $\sum \tau = 0$

$$m_1 g(x_1 - x_{cg}) + m_2 g(x_2 - x_{cg}) + m_3 g(x_3 - x_{cg}) + \ldots = 0$$

$$m_1 g x_1 + m_2 g x_2 + m_3 g x_3 + \ldots = (m_1 + m_2 + m_3 + \ldots) g x_{cg}$$

$$\rightarrow \boxed{x_{cg} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \ldots}{m_1 + m_2 + m_3 + \ldots} = \frac{\sum m_i x_i}{\sum m_i}}$$

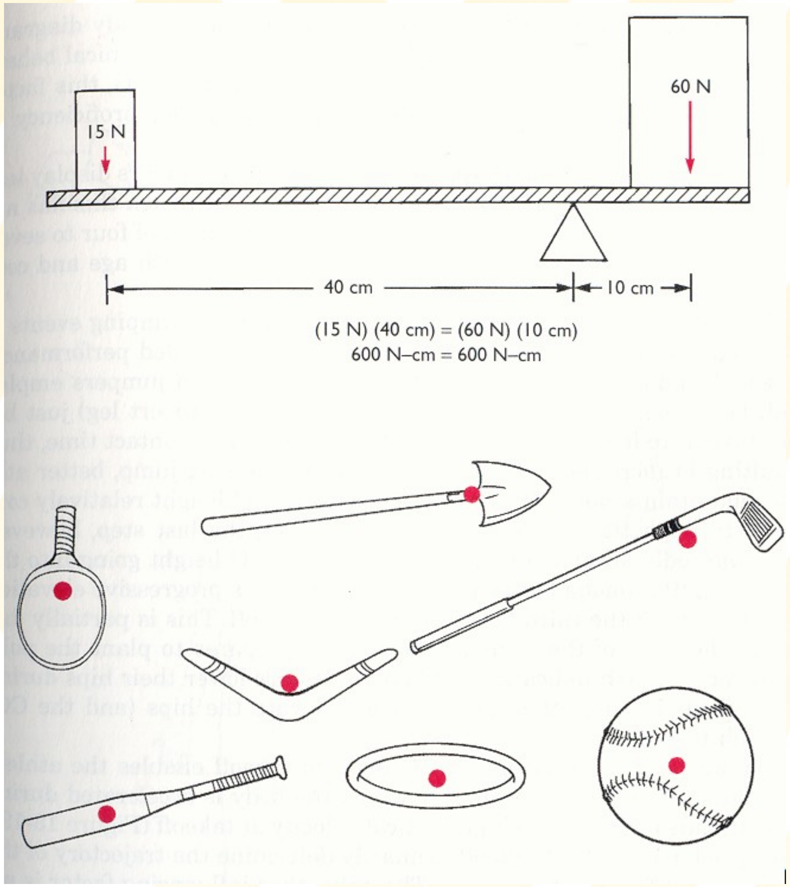
The same approach for the y- and z-component yields:

$$y_{cg} = \frac{\sum m_i y_i}{\sum m_i} \qquad z_{cg} = \frac{\sum m_i z_i}{\sum m_i}$$

Examples: Center of gravity

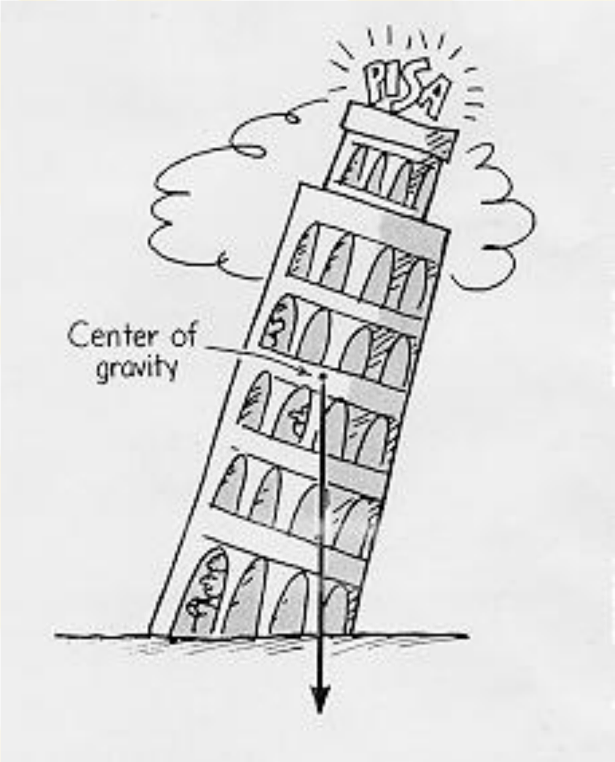
If the axis of rotation goes through the center of gravity, gravity will not cause any net torque on the object, i.e. it will not start to rotate from rest.

The center of gravity does not have to be part of the object itself.



Stability

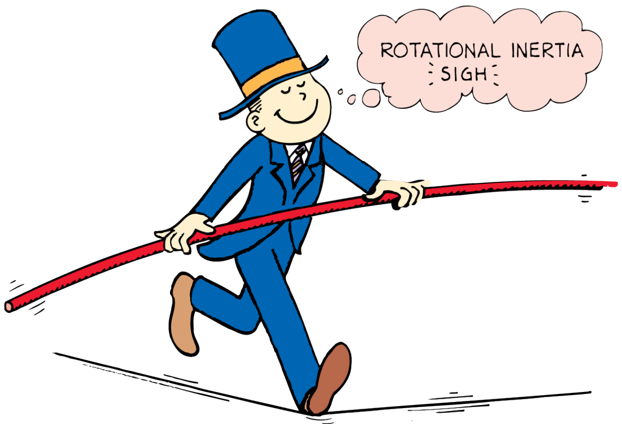
- If the CG of the object is above the area of support, the object will remain upright.
- If the CG is outside the area of support the object will topple.
- Leaning tower in Pisa, Italy - Center of gravity is still below its base at present

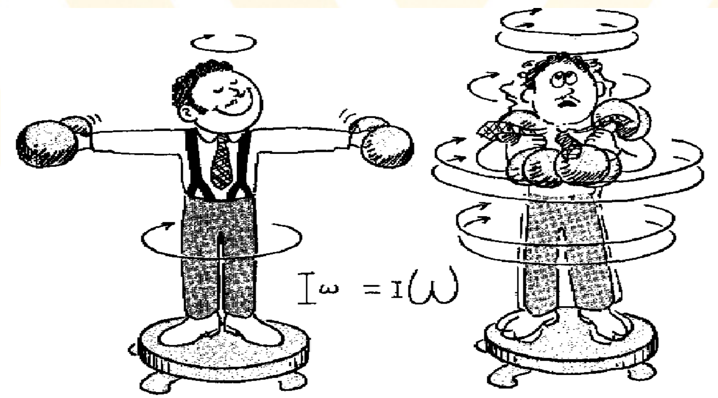


Today's lecture

Rotational Equilibrium and Dynamics:

- Moment of Inertia
- Rotational Kinetic Energy
- Angular Momentum



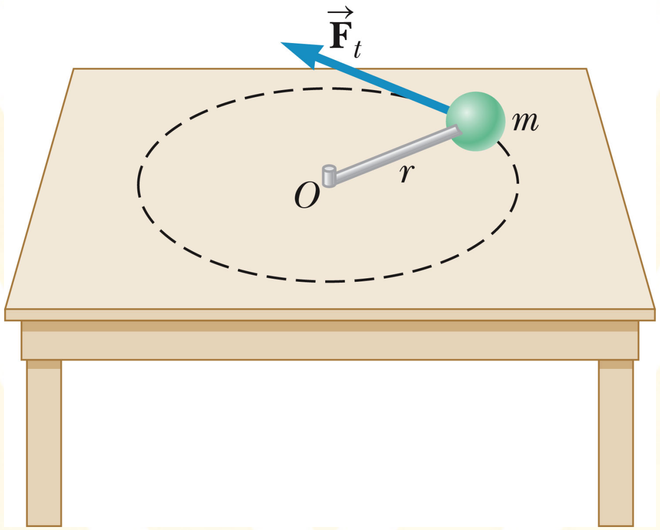


Torque vs. Angular Acceleration

When a rigid object is subject to a net torque, it undergoes an angular acceleration.

We will now derive an equation relating torque to angular acceleration. This is analogous to Newton's 2nd law (linear motion, F = ma):

An object of mass, m, is allowed to rotate around an axis through O (radius r). A tangential force, F_t, acts on the object causing a tangential acceleration, a_t, based on Newton's 2nd law:



$$F_t = ma_t$$

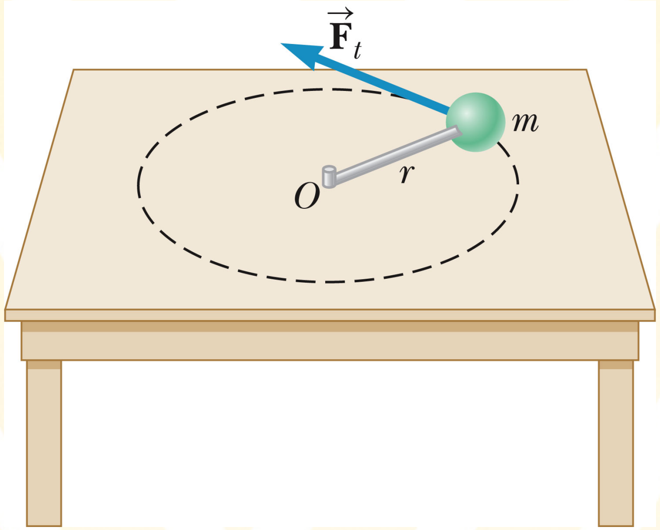
Multiplication with r and using $a_t = r \cdot \alpha$ yields:

$$F_t r = mra_t = mr^2\alpha$$

$$\rightarrow \quad \boxed{\tau = mr^2\alpha}$$

Newton's 2nd law & Rotation

$$\tau = mr^2\alpha$$



Definition of the Moment of Inertia of this system:

$$\boxed{I = mr^2}$$

Unit: kg m^2

The moment of inertia links torque with angular acceleration:

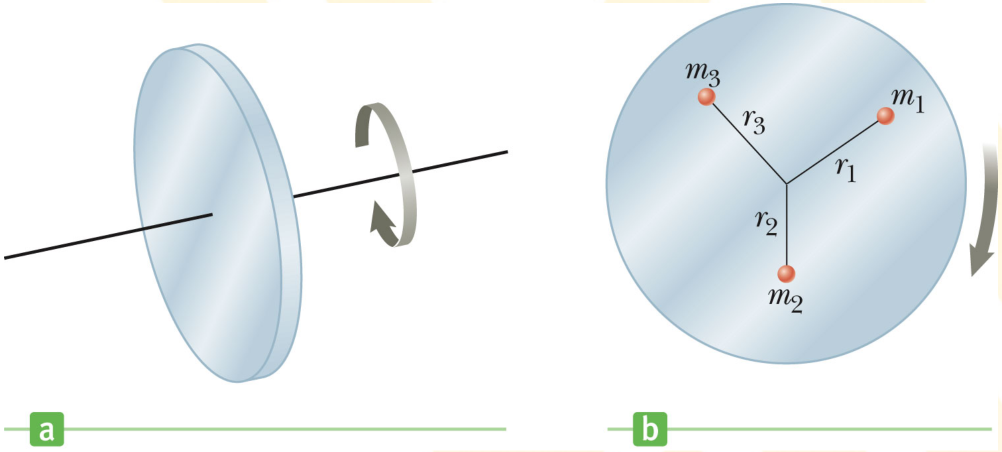
$$\boxed{\tau = I\alpha}$$

This is analogous to Newton's 2nd law for linear motion: F = ma

In case of rotational motion, the mass is replaced by the moment of inertia and the linear acceleration is replaced by the angular acceleration.

The angular acceleration is directly proportional to the net torque and inversely proportional to the moment of inertia.

Moment of Inertia



Until now we only discussed rotation of a point object. Now we discuss rotation of an extended rigid body.

The net torque on a disc rotating around an axis is the sum of all individual torques on all particles the disc consists of. As this is a rigid body, all particles have the same angular acceleration:

$$\sum \tau = \left(\sum mr^2\right)\alpha$$

Moment of Inertia: $\boxed{I = \sum mr^2 \quad \rightarrow \quad \sum \tau = I\alpha}$

Moment of Inertia

In contrast to linear motion, the distribution of an object's mass matters for rotation, i.e. it affects an object's moment of inertia:

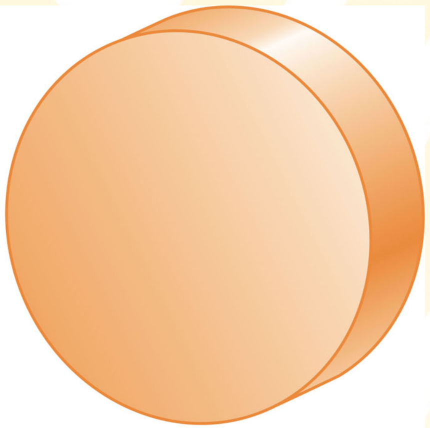
$$I = \sum mr^2 \qquad \sum \tau = I\alpha$$

If two objects have the same mass and radius, but different distributions of the mass, their moments of inertia will be different.

For example, the moment of inertia of object B will be higher compared to A, if both objects are of the same mass, since the mass is located at larger radii in case of B.

Consequently, it is more difficult to accelerate ring B.

A



B

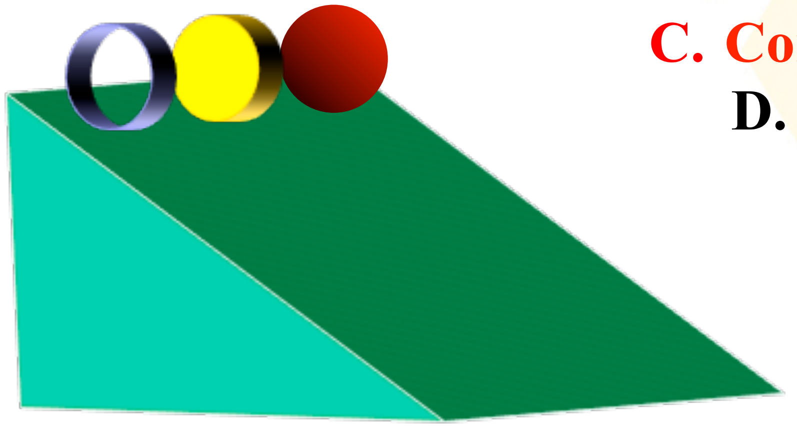
Clicker question

All shapes below have the same mass and same radius

In order to maximize the moment of inertia (I), the mass should be:

I= MR^2 1/2 MR^2 2/5 MR^2

Hoop Disk Sphere



A. Concentrated at the edges

B. Evenly distributed

C. Concentrated at the center

D. Makes no difference

Clicker question

All shapes below have the same mass and same radius

Since all of these shapes have the **same mass**, they all have the same force that will act to accelerate them down the incline (torque).

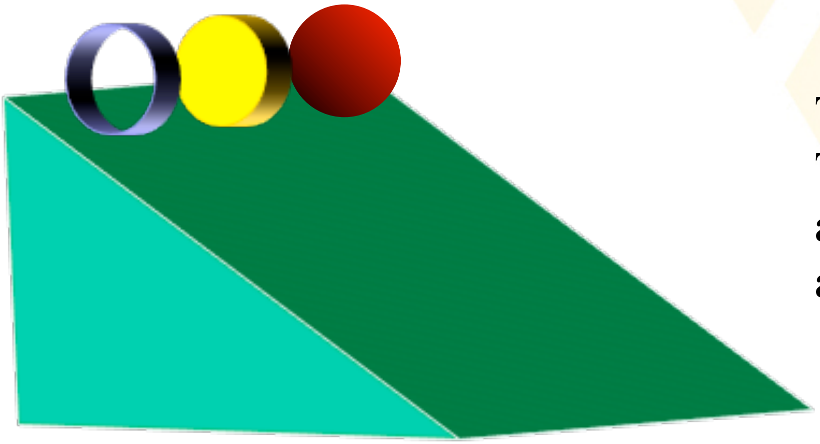
Which shape will win the race?

A. Hoop
B. Disk
C. Sphere
D. All same

I= MR^2 1/2 MR^2 2/5 MR^2

Hoop Disk Sphere

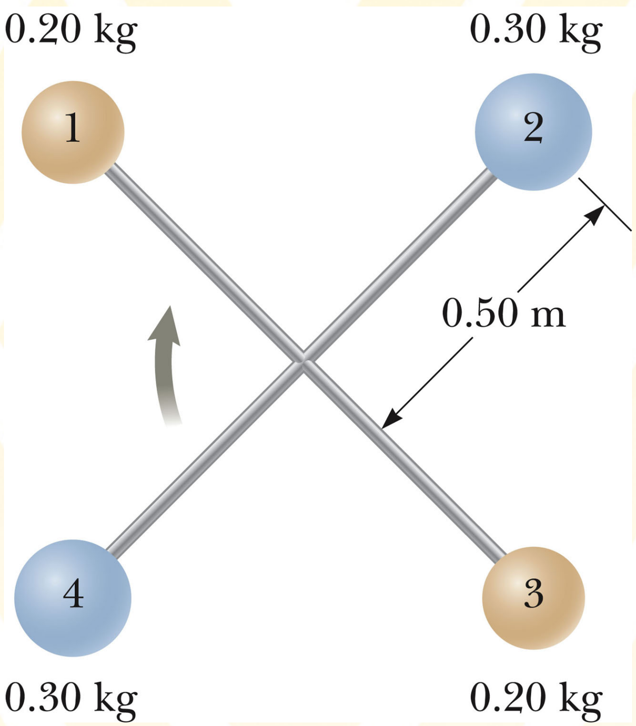
$$\Sigma\tau = I\alpha$$



They all have the same torque.
Thus, smaller I means a larger angular acceleration, which is also a larger linear acceleration.

Example problem: Moment of Inertia

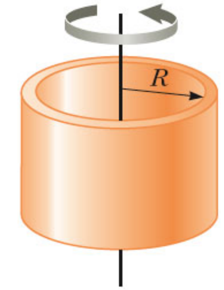
Calculate the moment of inertia of this baton made of four spheres about an axis perpendicular to the page and passing through the point where the rods cross.



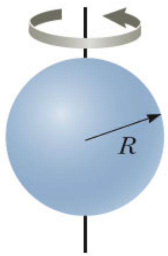
Moments of Inertia of more complex objects

Table 8.1 Moments of Inertia for Various Rigid Objects of Uniform Composition

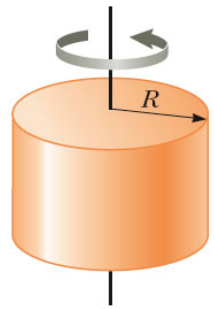
Hoop or thin cylindrical shell
$I = MR^2$



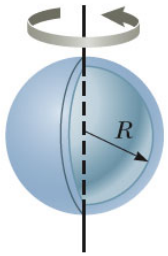
Solid sphere
$I = \frac{2}{5} MR^2$



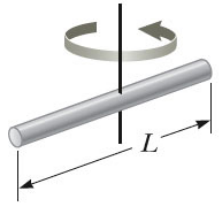
Solid cylinder or disk
$I = \frac{1}{2} MR^2$



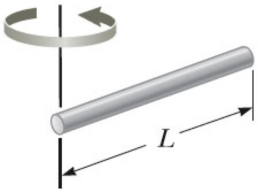
Thin spherical shell
$I = \frac{2}{3} MR^2$



Long, thin rod with rotation axis through center
$I = \frac{1}{12} ML^2$



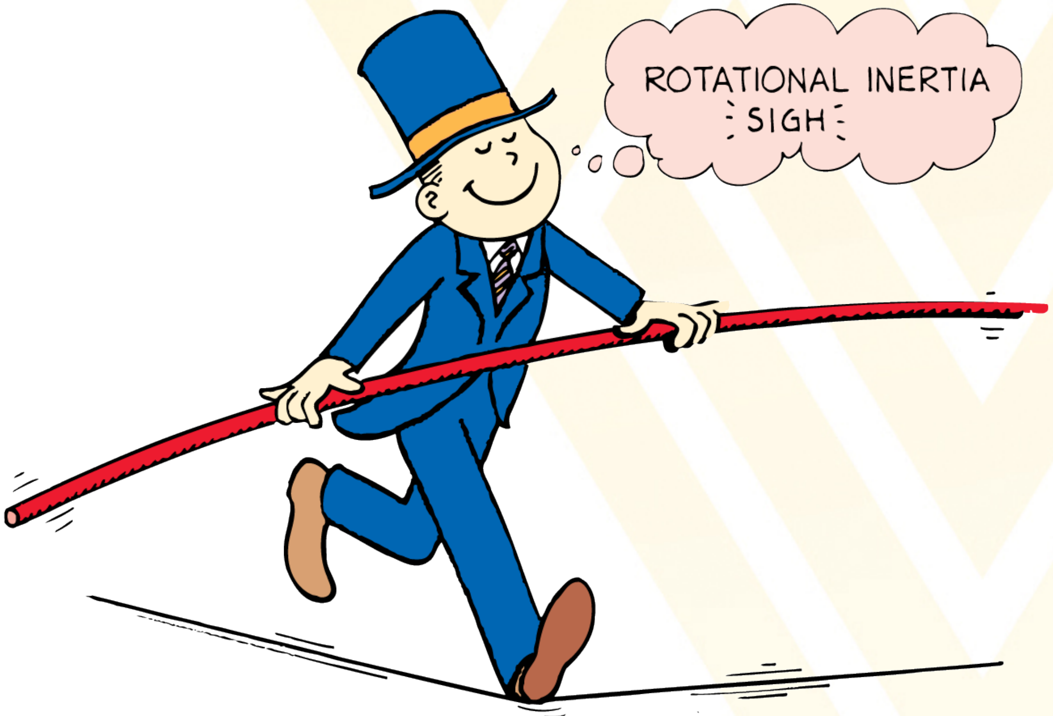
Long, thin rod with rotation axis through end
$I = \frac{1}{3} ML^2$



Rotational Inertia

Rotational Inertia

By holding a long pole, the tightrope walker increases his rotational inertia.



Rotational kinetic energy

We know how to calculate the kinetic energy of a particle of mass m moving at linear velocity, v:

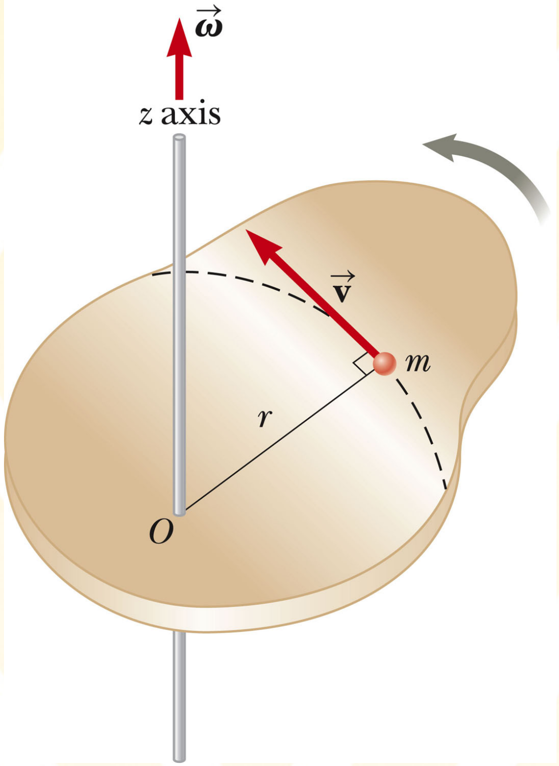
$$KE_t = \frac{1}{2}mv^2$$

Now, we calculate the kinetic energy of a thin plate rotating about an axis through O.

We have to sum up the kinetic energies of all particles the plate consists of:

$$KE_r = \sum(\frac{1}{2}mv^2) = \frac{1}{2}(\sum mr^2)\omega^2$$

$$\rightarrow \boxed{KE_r = \frac{1}{2}I\omega^2}$$



Conservation of Energy

In the absence of work done by non-conservative forces, rotational kinetic energy, KE_r, is the fourth type of energy we got to know in addition to translational kinetic energy, KE_t, gravitational potential energy, PE_g, and spring potential energy, PE_s.

In the absence of springs, the equation of conservation of energy is:

$$\boxed{(KE_t + KE_r + PE_g)_i = (KE_t + KE_r + PE_g)_f}$$

Example problem: Rotational kinetic energy

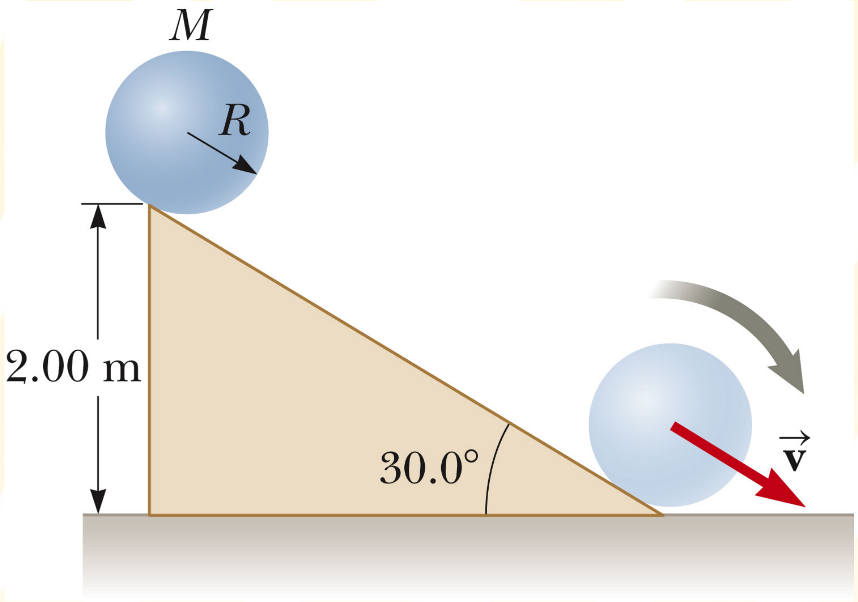
A ball of mass M and radius R starts from rest at a height of 2 m and rolls down an incline of 30 degree slope.

What is the linear speed of the ball when it leaves the incline?

Assume that the ball rolls without slipping.

Moment of Inertia for a solid sphere:

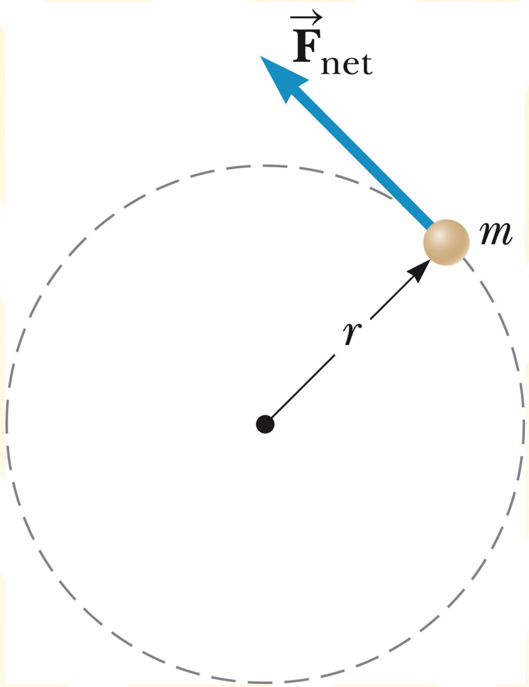
$$I = \frac{2}{5} M R^2$$



Angular Momentum

The force acting on the system shown on the right results in a net torque:

$$\sum \tau = I\alpha = I\frac{\Delta\omega}{\Delta t} = \frac{I\omega_f - I\omega_i}{\Delta t}$$



Now we introduce the angular momentum (linear momentum p = mv):

$$\boxed{L = I\omega}$$

Substituting this into the first equation yields:

$$\boxed{\sum \tau = \frac{L_f - L_i}{\Delta t} = \frac{\Delta L}{\Delta t}}$$

This equation is analogous to Newton's 2nd law for linear motion: $F = \dfrac{\Delta p}{\Delta t}$

If there is no net torque, $L_f = L_i$ and angular momentum is conserved.

Summary of conservation laws

Now, we know three fundamental conservation laws in physics:

1. Conservation of mechanical energy, if there is no work done by non-conservative forces:

$$(KE_t + KE_r + PE_g + PE_s)_i = (KE_t + KE_r + PE_g + PE_s)_f$$

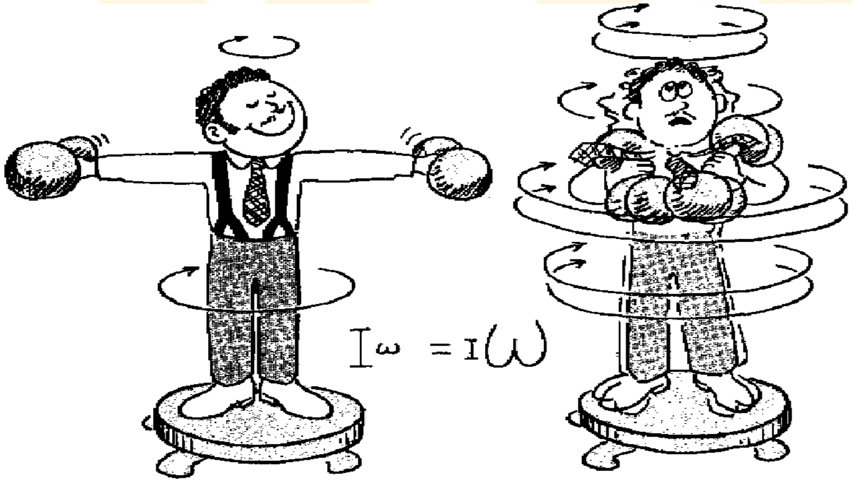
2. Conservation of linear momentum, if there is no net external force on the system (two objects):

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

3. Conservation of angular momentum, if there is no net torque on the system:

$$I\omega_i = I\omega_f$$

Applications of conservation of angular momentum



A figure skater usually spins in the finale of his act. He/she can change his angular velocity by stretching his/her arms out.

Why does this work? - It's conservation of angular momentum: There is no net torque. By stretching out his/her arms, the skater increases his moment of inertia. Consequently his angular velocity must decrease.

$$I = \sum mr^2 \qquad\qquad L_f = I_f\omega_f = L_i = I_i\omega_i$$

Relations

Linear Motion	Rotational Motion
Mass m	Moment of Inertia I
Linear velocity $\mathbf{v}$	Angular velocity ω
Translational KE $\frac{1}{2}mv^2$	Rotational KE $\frac{1}{2}I\omega^2$
Linear momentum $\mathbf{p} = m\mathbf{v}$	Angular momentum $L = I\omega$
$\vec{F}_{net} = \sum \vec{F} = \frac{\Delta \vec{p}}{\Delta t}$	$\tau_{net} = \sum \tau = \frac{\Delta L}{\Delta t}$

(F=ma)
when m constant

Note: if I is constant, $\frac{\Delta L}{\Delta t} = \frac{I\omega - I\omega_o}{\Delta t} = \frac{I\Delta\omega}{\Delta t} = I\alpha$

Summary

- Definition of the Moment of Inertia of a system: $I = \sum mr^2$ Unit: kg m^2

- The moment of inertia links torque with angular acceleration: $\tau = I\alpha$

- Rotational kinetic energy: $KE_r = \frac{1}{2}I\omega^2$

- Angular momentum: $L = I\omega$ $\sum \tau = \frac{L_f - L_i}{\Delta t} = \frac{\Delta L}{\Delta t}$

- Angular momentum will be conserved, if there is no net torque on the system.

- The moment of inertia can be changed by rearranging the distribution of mass of an object. In the absence of a net torque, this will change the angular velocity.

$$I = \sum mr^2 \qquad L_f = I_f\omega_f = L_i = I_i\omega_i$$