1-10 Prove or find a counter example. Do one of each pair of questions. Circle the one you will be proving. (Note for solutions – original paper had the problems paired off two to a page (1-2, 3-4, 5-6, 7-8, and 9-10)

1) The square of any integer has the form 4k or 4k+1 for some integer k.
Proof (by cases):
Suppose n is an integer. We know n is either even or odd.
Case 1: n is even.
By definition of even n = 2j for some integer j. Thus n² = (2j)² = 4j². But j² is an integer since it is the product of integers. Call this integer k. Thus n² = 4k.
Case 2: n is odd.
By definition of odd n = 2j + 1 for some integer j. Thus n² = (2j+1)² = 4j² + 4j + 1 = 4(j² + j) + 1. But j² + j is an integer since it is the sum and product of integers. Call this integer k. Thus n² = 4k + 1. QED.

2) If n² is even then n is even.
Proof: (by Contraposition). We shall show if n is not even (odd) then n² is not even (odd).
Suppose n is odd. Then by definition of odd we know n = 2k + 1 for some integer k. Thus n² = (2k+1)² = 4k² + 4k + 1 = 2(2k² + 2k) + 1. But 2k² + 2k is an integer since it is the sum and product of integers. Thus n² = 2(an integer) + 1 and by definition is odd. QED.

3) For all integers a, b and c, if ab| c then a|c and b |c.
Proof: Suppose a, b, and c are integers and ab|c. [We must show a|c and b|c]. Since ab|c by definition of divides we know c = ab*k for some integer k. By the associative law this means c = a*(bk) and c = b*(ak). But bk and ak are integers since they are the product of integers. Thus c = a*(an integer) and c = b*(an integer). Therefore by definition of divides a | c and b | c. QED.

4) A necessary condition for an integer to be divisible by 6 is that it be divisible by 3.
Proof. [We must show that if integer is divisible by 6 then it is divisible by 3].
Let n be an integer divisible by 6. By definition n = 6k for some integer k. Thus n = 2 * 3*k = 3*(2k). But 2k is an integer since it is the product of integers. Thus n = 3*(an integer). Hence by definition of divides 3 | n. QED

5) For all real numbers x, \( \lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil \)
False: Counter example: \( \lceil 1/2 + 1/2 \rceil = \lceil 1 \rceil = 1 \) but \( \lceil 1/2 \rceil + \lceil 1/2 \rceil = 1 + 1 = 2 \)

6) For all real number x and y \( \lceil xy \rceil = \lceil x \rceil \cdot \lceil y \rceil \)
False: Counter example: \( \lceil 1/2 \cdot 1/2 \rceil = \lceil 1/4 \rceil = 1 \) but \( \lceil 1/2 \rceil \cdot \lceil 1/2 \rceil = 1 \cdot 0 = 0 \)
7) Prove $\sqrt{2}$ is irrational (you may assume problem #2)

Proof (by contradiction):

Suppose $\sqrt{2}$ is rational. Then by definition $\sqrt{2} = m/n$ for some integers $m$ and $n$ where $n \neq 0$. Since we can cancel common factors we may assume $m$ and $n$ have no common factors. Now $\sqrt{2} = m/n$ implies $\sqrt{2} \cdot n = m$. Squaring both sides we obtain $2n^2 = m^2$. But $n^2$ is an integer since it is a product of integers. Thus $2n^2 = m^2$ implies $m^2$ is even. By problem #2 this implies $m$ is also even. Now by definition of even $m = 2k$ for some integer $k$. Thus $m^2 = 4k^2$. Substituting this into the equation $2n^2 = m^2$ we obtain $2n^2 = 4k^2$, and by simplifying we obtain $n^2 = 2k^2$. But $k^2$ is an integer since it is the product of integers. Hence by definition of even $n^2$ is even, and by problem #2 we know $n$ is even. Contradiction $2 \mid n$ and $2 \mid m$ but $m$ and $n$ have no common factors. QED.

8) Show there is no greatest negative rational number.

Proof. (by contradiction)

Suppose not. Suppose there is a greatest negative rational number. Call it $n$. Now consider $n/2$. Since $n < 0$ we know $n + n < n + 0$. Thus $2n < n$ and $n < n/2$. Thus $n/2$ is greater than $n$. Also since $n < 0$ we know $n/2 < 0/2$ so $n/2 < 0$. Thus $n/2$ is a negative number greater than $n$. In order to get a contradiction it only remains to show $n/2$ is rational. We know $n$ is rational so $n = a/b$ for some integers $a$ and $b$ with $b \neq 0$. Hence $n/2 = a/(2b)$. But $2b$ is an integer since it is the product of integers and $2b \neq 0$ by the zero product property since neither 2 nor $b$ are zero. Let $c$ be the non-zero integer $2b$. Thus $n/2 = a/c$ and by definition $n/2$ is rational. Contradiction.

There is no greatest negative rational number.

Prove 9 or 10 by induction:

9) $\sum_{i=1}^{n} i^3 = \left[ \frac{n(n+1)}{2} \right]^2$, for all integers $n \geq 1$.

Proof: (By induction)

Basis Step: When $n = 1$ the LHS is $1^3 = 1$ and the RHS is $\left[ \frac{1 \cdot 2}{2} \right]^2 = 1$ so the statement is true in this case.

Inductive step: Suppose $\sum_{i=1}^{k} i^3 = \left[ \frac{k(k+1)}{2} \right]^2$ for some integer $k \geq 1$. [We must show

$\sum_{i=1}^{k+1} i^3 = \left[ \frac{(k+1)(k+2)}{2} \right]^2$. By adding $(k+1)^3$ to both sides of our hypothesis we obtain

$\sum_{i=1}^{k+1} i^3 = \left[ \frac{(k+1)(k+2)}{2} \right]^2 + (k+1)^3$. Simplifying, we see the LHS $= \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} = (k+1)^2 \cdot \left( \frac{k^2}{4} + \frac{4(k+1)}{4} \right) = (k+1)^2 \cdot \left( \frac{k^2 + 4k + 4}{4} \right) = (k+1)^2 \left( \frac{(k+2)^2}{4} \right) = \left[ \frac{(k+1)(k+2)}{2} \right]^2$. Thus $\sum_{i=1}^{k+1} i^3 = \left[ \frac{(k+1)(k+2)}{2} \right]^2$. QED.
10) \(n^3 - 7n + 3\) is divisible by 3 for all integers \(n \geq 0\).
Proof: (by induction)
Basis step: When \(n = 0\) we see that \(n^3 - 7n + 3 = 3\) and \(3|3\) so the statement is true in this case.
Inductive step. Suppose \(k^3 - 7k + 3\) is divisible by \(k\) for some integer \(k \geq 0\). Then by definition \(k^3 - 7k + 3 = 3p\) for some integer \(p\). [We must show \((k+1)^3 - 7(k+1) + 3\) is divisible by 3].
Now \((k+1)^3 - 7(k+1) + 3 = k^3 + 3k^2 + 3k + 1 - 7k + 3 = (k^3 - 7k + 3) + 3k^2 + 3k - 6\). By substituting in the inductive hypothesis and simplifying we have \((k+1)^3 - 7(k+1) + 3 = 3p + 3k^2 + 3k - 6 = 3(p + k^2 + k - 2)\). But \(p + k^2 + k - 2\) is an integer since it is the sum and product of integers. Call this integer \(q\). Thus \((k+1)^3 - 7(k+1) + 3 = 3q\) where \(q\) is an integer and by definition \(3 |{(k+1)^3 - 7(k+1) + 3}\). QED

11) One urn contains one blue ball (B) and three Green balls (G1, G2, G3). A second urn contains one red ball (R) and two blue balls (B1 and B2). An experiment is performed in which one of the two urns is chosen at random and then two balls are randomly chosen from it, one after the other without replacement.

a) What is the total number of outcomes of this experiment?
\(N(\text{picking the first urn}) + N(\text{picking the second urn})\) (since you pick \(U_1\) OR \(U_2\))
\[4 \times 3 + 3 \times 2 = 12 + 6 = 18\]

b) What is the probability that two blue balls are chosen?
Only way to get two blue balls is to choose Urn 2 then \{B1,B2\} or \{B2, B1\}
\[P(\text{two blue balls}) = \frac{2}{18}\]

12) a) How many ways can the letters of the word NUMERICAL be arranged in a row? \(9!\)

b) How many ways can the letters of the word NUMERICAL be arranged in a row if NUM must remain together (in order) as a unit? \(7!\)

13) a) How many 5 digit integers (integers from 10,000 to 99,999) are divisible by 3 or 5?
\(1000 = 2000 \times 5 \ldots \ldots \ldots 99,995 = 19,999 \times 5\); \(N(\text{divisible by 5}) = 19,999 - 2000 + 1 = 18,000\)
\(10,002 = 3 \times 3334 \ldots \ldots 99,999 = 3 \times 33,333\); \(N(\text{divisible by 3}) = 33,333 - 3334 + 1 = 30,000\)
\(10,005 = 667 \times 15 \ldots \ldots 99,990 = 6666 \times 15\); \(N(\text{divisible by 15}) = 6666 - 667 + 1 = 6,000\)
\(N(\text{divisible by 3 or 5}) = 18,000 + 30,000 - 6,000 = 42,000\)

b) How many 5 digit integers CONTAIN the digit 3 exactly once?
METHOD 1:
\(N(\text{numbers from 1 to 99,999 that contain 3 exactly once}) - N(\text{numbers from 1 to 9,999 that contain 3 exactly once})\)

\[
\begin{align*}
3 & \_ \_ \_ \_ \_ \quad 3 \text{ can go 5 places with 9 choices for other 4 numbers } 5 \times 9^4 \\
3 & \_ \_ \_ \_ \quad 3 \text{ can go 4 places with 9 choices for the other 3 numbers } 4 \times 9^3
\end{align*}
\]
Answer \(5 \times 9^4 - 4 \times 9^3 = 9^3(45 - 4) = 9^3(41) = 29,889\)
METHOD 2:
N(numbers from 10,000 to 99,999 that contain 3 exactly once) = N(ways 3 is first) + N(ways 3 is second) + N(ways 3 is third) + N(ways 3 is fourth) + N(ways 3 is fifth).
Keeping in mind 0 can not be in the first place this is
\[9^4 + 8 \times 9^3 + 8 \times 9^3 + 8 \times 9^3 + 8 \times 9^3 = 9^4 + 32 \times 9^3 = 9^4(41) = 29,889\]

c) How many 5 digit integers CONTAIN the digit 3 at least once?
METHOD 1:
N(numbers from 1 to 99,999 that contain 3 at least once) − N(numbers from 1 to 9,999 that contain 3 at least once).
But we know: N(numbers from 1 to 99,999 that contain 3 at least once) = 100,000 − N(numbers from 1 to 99,999 that don’t contain 3) = 100,000 − 9^5 = 40951. Similarly,
N(numbers from 1 to 9,999 that contain 3 at least once = 10,000 − 94 = 3439. So,
N(numbers from 10,000 to 99,999 that contain 3 exactly once) = 40951 − 3439 = 37,512

METHOD 2:
N(contain the digit at least once) = (99,999 − 10000 + 1) − N(don’t contain 3) − 90,000 − N(don’t contain 3). To compute N(don’t contain 3) we keep in mind that the first term cannot be 0 and no digit can be 3. Thus N(don’t contain 3) = 8*9*9*9*9 =52,488.
Thus N(contain the digit at least once = 90,000 − 52,488 = 37,512

14) A group of 7 people are attending the movies together. Two of the people do not want to sit side-by-side. How many ways can the seven be seated together in a row?
N( seated with AB not side by side) = N(7 can be seated) - [N(seated with AB) + N(seated with BA)] = 7! − [6! + 6!] = 7! − 2*6! = 3600

15) Barney Rubble is a section chief for an electric utility company. The employees in his section cut down tall trees, climb poles, and splice wire. Rubble reported the following information to the management of the utility: Of the 100 employees in my section, 45 can cut tall trees, 50 can climb poles, 57 can splice wire, 28 can cut trees and climb poles, 20 can climb poles and splice wire, 25 can cut trees and splice wire, 25 can cut trees and splice wire, 25 can cut trees and splice wire, 11 can do all three

How many can’t do any (management trainees)? 10 (see diagram)