Section 3.1: Functions

In Chapter 2, we talked about how equations with the variables x and y give a relationship between them: as x changes, it means that y must change too in order to keep the equation true. Here, we will describe a similar, special type of relationship, called a function.

The key distinction about functions are that they take in an input and produce an output. For example, consider the process of mailing a package. Based on the weight of the package (and possibly other factors), there is a cost of mailing the package. We can think of this as a function, where the weight of the package is the input, and the cost is the output. In this way, you can think of a function as a sort of factory: we have some raw material going in, and the function transforms it into something else, which is the output we get.

For our purposes, the inputs and outputs will be numbers, variables, or algebraic expressions.

Example 1. Consider the function that takes a number as input and squares it. So if we put in the number 5 as input, we get the number 25 as output. If we put in the number 9 as input, we get the number 81 as output. We can use a table to more efficiently express a bunch of inputs and outputs.

Input	9	5	2	0	-3
Output	81	25	4	0	9

An even more efficient way of representing functions will be to use letters. That is, we can let "f" be the function in the example of taking a number as input and squaring it. We use the notation "$f(5)$" to mean "apply the function f using the input 5". That is, $f(5) = 25$, and $f(9) = 81$.

Furthermore, we can use a variable to represent the input. That is, if x is any input number, then we can express the rule of the function by saying that $f(x) = x^2$. This is how to express a function with a formula, and it will be the way we'll write functions most often.

We still haven't given the whole definition of what a function is, because there's one more important ingredient, which is that when we put in the same input, we must always get the same output. This is the key difference between a relationship defined by a function and a relationship defined by an equation, as we'll explore more later. This gives us our formal definition.

Definition. A *function* f is a rule that takes inputs from a set, called the *domain*, and to each input it assigns EXACTLY one output. The set of outputs achieved is called the *range*.

Example 2. Let x represent an input and let y represent an output. Consider the table of inputs and outputs below.

x	1	2	2	3	4
y	1	2	4	0	9

This table does NOT represent y as a function of x, because the input of 2 has more than one output.

Therefore, whenever we express inputs and outputs with tables (or graphs, as we'll see in 3.2), we have to check to make sure the same input isn't repeated with different outputs to be sure the table actually represents a function.

Sometimes we will want to define a function by a combination of formulas, depending on what the input is. This is called a *piecewise defined function*.

Example 3. We define a piecewise defined function f by

$$f(x) = \begin{cases} x+11 & \text{if } x < -1 \\ x^2+2 & \text{if } x \geq -1 \end{cases}$$

Thus, the rule changes for different inputs. What is $f(0)$? $f(-3)$? $f(4)$? $f(-1)$?

Answer. Since the input 0 is greater than or equal to -1, we use the second rule. Therefore, $f(0) = (0)^2 + 2 = 2$.
Since the input -3 is strictly less than -1, we use the first rule. Therefore, $f(-3) = (-3) + 11 = 8$.
Since the input 4 is greater than or equal to -1, we use the second rule. Therefore, $f(4) = (4)^2 + 2 = 18$.
For the input of -1, we have to be careful, since this is exactly the point where the rule changes. Since -1 is equal to -1, we use the second rule. Therefore, $f(-1) = (-1)^2 + 2 = 3$.

We can also evaluate functions using algebraic expressions as inputs.

Example 4. Define a function f by $f(x) = x^2 - x$. Evaluate

(a) $f(-a)$
(b) $f(a+1)$
(c) $f(a^3)$

Answer. Here, a is being used as a placeholder to represent any number.

(a) $f(-a) = (-a)^2 - (-a) = a^2 + a$
(b) $f(a+1) = (a+1)^2 - (a+1) = (a^2+2a+1) - (a+1) = a^2+2a+1-a-1 = a^2+a$
(c) $f(a^3) = (a^3)^2 - (a^3) = a^6 - a^3$

An important part of the definition of a function was the set of possible inputs, which we call the domain. Since we will be focused on functions whose inputs are real numbers, we will always assume that the domain of a function is the largest set of real numbers that can be used as an input to produce a real number as an output.

Therefore, we have two major things to look out for so far:

(1) If an input makes a denominator in the function equal to zero, it is NOT in the domain.
(2) If an input causes the function to take a square root (or other even order root) of a negative number, it is NOT in the domain.

Example 5. Consider the function $f(x) = \dfrac{1}{x+3}$. Then inputting the value of -3 is not allowed, since it would make the denominator zero. Any other input would work, so we can express the domain as $(-\infty, -3) \cup (-3, \infty)$, or as $\{x \mid x \neq -3\}$, or we can use a shorthand and just say that $x \neq -3$.

Example 6. Consider the function $g(x) = \sqrt{x-5}$. Then since we aren't allowed to take the square root of a negative number, the domain is going to be the set of all values of x such that $x - 5 \geq 0$. This means that $x \geq 5$, so we can express the domain of g in interval notation as $[5, \infty)$.

SUMMARY:

- A function is a rule that takes each input value in the domain to exactly one output value.
- The domain is the largest set of real number inputs for which the function produces real number outputs.
- We have multiple ways of representing functions:
  (1) With words: "add 1 to the input and square the result"

  (2) With a table of a few inputs:

x	-2	-1	0	1.5	5
$f(x)$	1	0	1	6.25	36

  (3) With a formula: $f(x) = (x+1)^2$

  (4) With a graph: (This is the subject of Section 3.2!)