## Section 3.2: Graphs of Functions

Most of the basics of graphing was already covered in Chapter 2, when we learned how to graph equations. To graph a function $f$, we just graph the equation $y = f(x)$ for all values of $x$ in the domain of $f$. That is, the graph of $f$ is the collection of all points $(x, f(x))$ where $x$ is in the domain.

**Example 1.** (Graphs we know)

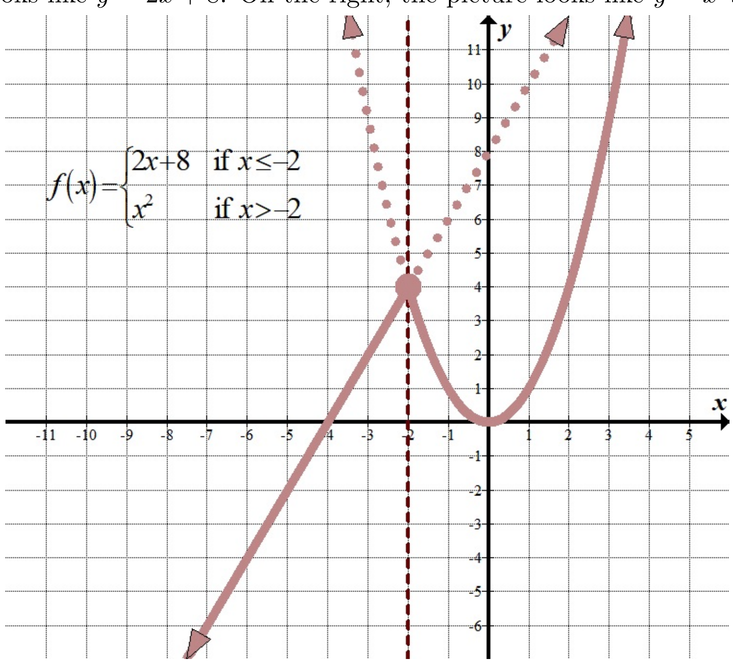
- We saw that the equation $y = mx + b$ has a graph that is a line with slope $m$ and $y$-intercept $b$. Therefore, the function $f(x) = mx + b$ has the same graph. We call it a *linear function*.
- A special kind of linear function occurs when the slope is 0, so the line is horizontal and has equation $y = b$. Therefore, the function $f(x) = b$ has this horizontal line as its graph. We call it a *constant function* because the output values are the same no matter what input value we use.
- More complicated functions can be graphed by plotting points, as we did in Section 2.1. (Take a look at Example 7 and Example 8 in Section 3.2 to see examples of graphing functions by plotting points.)

Graphing piecewise defined functions is a little trickier, because we have to change the rule depending on where along the $x$-axis we are.

**Example 2.** Graph the piecewise defined function given by

$$f(x) = \begin{cases} 2x + 8 & \text{if } x \leq -2 \\ x^2 & \text{if } x > -2 \end{cases}.$$

**Answer.** We know how to graph the line $y = 2x + 8$, and we could plot points to graph $y = x^2$. However, the rule of the function changes at $x = -2$. We can think of the vertical line $x = -2$ as splitting the graph into two parts. On the left, the picture looks like $y = 2x + 8$. On the right, the picture looks like $y = x^2$.

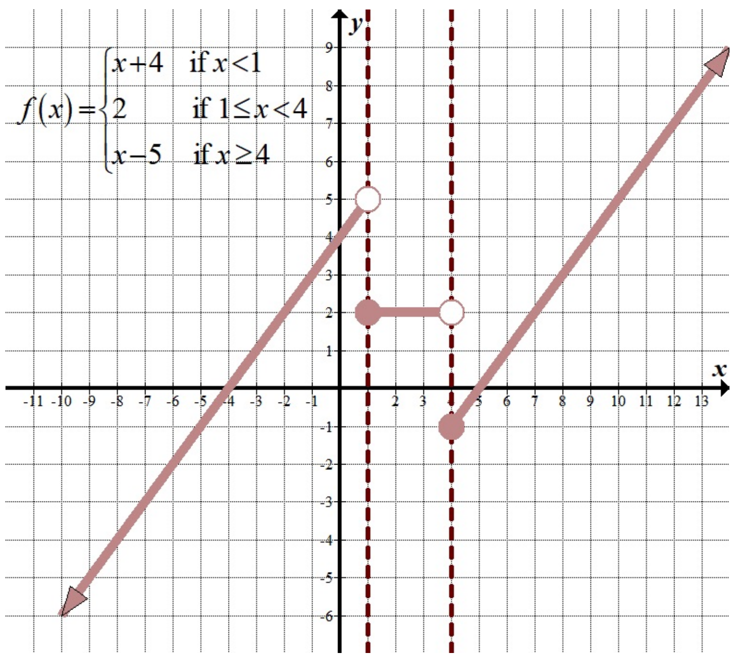


A special type of piecewise defined function is one in which all the different rules used are linear. This is called a *piecewise linear function.*

**Example 3.** Graph the piecewise linear function given by

$$f(x) = \begin{cases} x+4 & \text{if } x < 1 \\ 2 \text{ if } 1 \leq x < 4 & \\ x-5 & \text{if } x \geq 4 \end{cases}.$$

**Answer.** Now, the rule of the function changes at both $x = 1$ and at $x = 4$. On the interval $(-\infty, 1)$, the graph looks like $y = x + 4$. On the interval $[1, 4)$, the graph looks like $y = 2$. On the interval $[4, \infty)$, the graph looks like $y = x - 5$. We use closed dots to indicate a point that is on the graph, and we use open circles to indicate that a point is NOT on the graph.

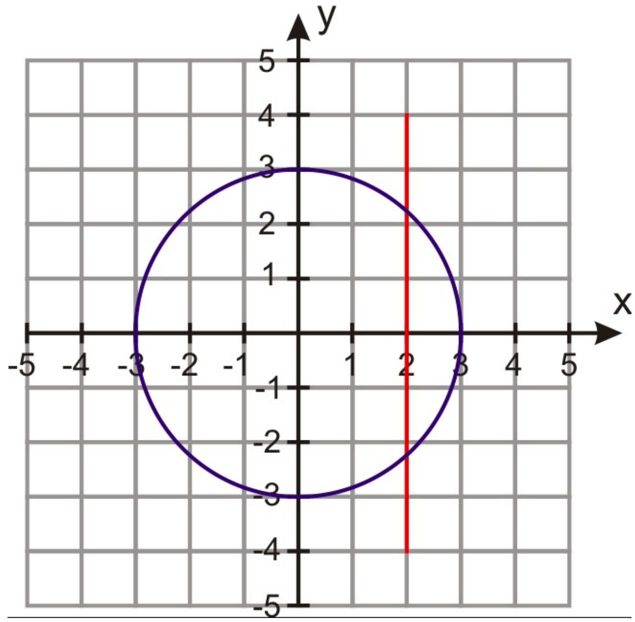



Note the open circles at the points $(1, 5)$ and $(4, 2)$, compared with the closed dots at the points $(1, 2)$ and $(4, -1)$. The reason that $(1, 2)$ is included in the graph and $(1, 5)$ is not is because function says that when we input $x = 1$, we get the output from the second rule, so the output is 2. This is similarly the case when $x = 4$. You will have more practice graphing piecewise linear functions on the homework.

Now that we know that the graph of every function gives a curve in the $xy$-plane, we can ask whether the reverse is true. That is, we can ask whether every curve in the $xy$-plane is the graph of a function.

The answer is no! Our definition of a function said that for every value of $x$ in the domain, there is exactly one output value $f(x)$. Therefore, if a curve contains two or more points that have the same $x$ value but different $y$ values, we know that this curve can NOT be the graph of a function.

**Example 4.** Consider the curve shown below. This circle is the graph of the equation $x^2 + y^2 = 9$.



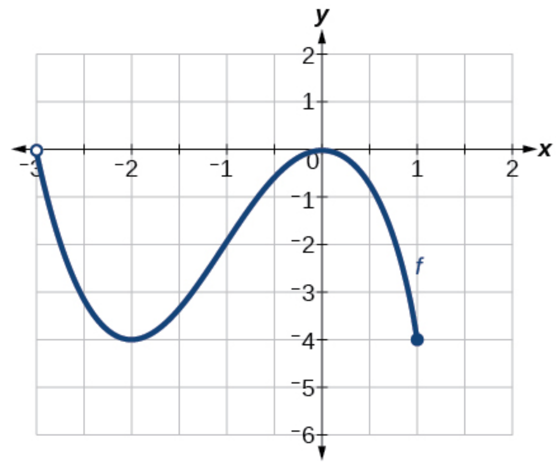

Then this curve has points at both $(0, 3)$ and at $(0, -3)$. Those points have the same $x$ value, but different $y$ values. That tells us that this can't be the graph of a function. So it's important to note that even though this is the graph of an *equation*, it is NOT the graph of a *function*.

So we have an easy way of checking whether a curve in the $xy$-plane is actually the graph of a function. If there are any vertical lines that intersect the curve at two or more points, then the curve is not the graph of a function. We can see this test in the picture above. The line $x = 2$ (shown in red) intersects the circle at two points, so we know the circle is not the graph of a function. This test is known as the Vertical-Line Test.

**Vertical-Line Test:** A curve in the $xy$-plane is the graph of a function if and only if no vertical line intersects the curve at more than one point.

If a curve IS the graph of a function, we want to be able to use that graph to tell us information about the function. Since the graph represents when $y = f(x)$, we can determine function values by considering the $y$-coordinates of points on the graph. Similarly, using the graph, we can determine the domain and range of the function. The domain is the set of all possible $x$-coordinates of points on the graph, and the range is the set of all possible $y$-coordinates of points on the graph.

**Example 5.** Consider the following graph of a function $f$.




Evaluate $f(-2)$, $f(0)$, and $f(-1)$, and find the domain and range of $f$.

**Answer.** To find the value of $f(-2)$, we must find the $y$-coordinate of the point on the graph whose $x$-coordinate is $-2$. So we move along the $x$-axis to $x = -2$. Moving down to find the point on the graph, we find that the point $(-2, -4)$ is on the graph. Therefore, we have that $f(-2) = -4$.

Similarly, to find $f(0)$, we look at $x = 0$ on the $x$-axis. The point on the graph with $x$-coordinate 0 is the point $(0, 0)$. This means that $f(0) = 0$.

Sometimes we will need to estimate, which is the case for $f(-1)$. At $x = -1$, it looks like the point on the graph has $y$-coordinate that is roughly $-2$, although it might be slightly different. For our purposes, it is close enough that we can say that $f(-1) = -2$.

We can find the domain by looking at which values of $x$ are used in the graph. Note that there is an open circle at $x = -3$, which means that value of $x$ is not included in the domain. On the right, the graph goes up to and including $x = 1$. Therefore, the domain of the function is the interval $(-3, 1]$.

For the range, we do the same thing except that we look at which values of $y$ are used in the graph. So now, we're looking up and down instead of looking left to right. The smallest value of $y$ that is used on the graph is $y = -4$. The largest value of $y$ that is used on the graph is $y = 0$. Note that even though we have an open circle at $(-3, 0)$, we also have the point $(0, 0$ which IS included in the graph, which means that $y = 0$ IS included in the range. Therefore, the range of the function is $[-4, 0]$.

SUMMARY:

- The graph of a function is the set of all points in the $xy$-plane such that $y = f(x)$.

- To sketch the graph of a function, we can use the same techniques we used to graph equations.

- A curve in the $xy$-plane can only be the graph of a function if it passes the Vertical-Line Test.

- From the graph of a function we can determine:
  (1) Function values
  (2) Domain of the function
  (3) Range of the function