Section 3.4: Quadratic Functions and Applications

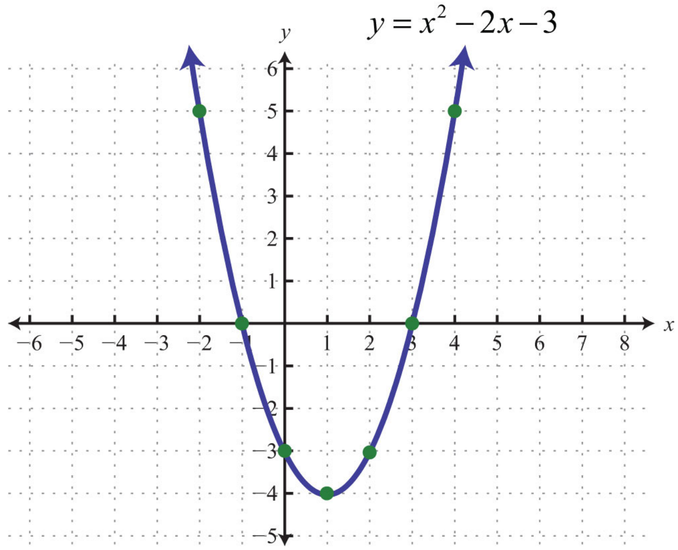
In this section, we'll learn more about quadratics, and analyze their graphs.

Definition. A *quadratic function* is a function whose rule is given by a polynomial of degree 2. We can write any quadratic function in the form $f(x) = ax^2 + bx + c$, where a, b, and c are numbers, and $a \neq 0$.

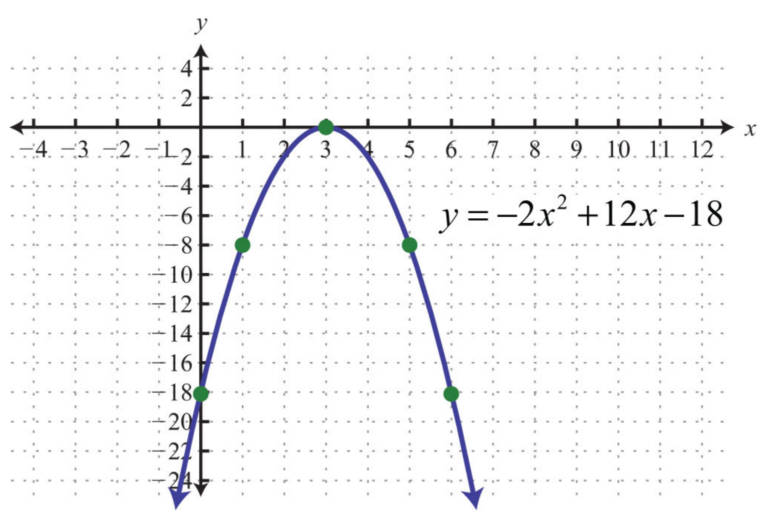
In Chapter 2, we did a couple examples where we graphed quadratic equations, and we may have noted the shape of these graphs.

Example 1. Graph the quadratic functions $f(x) = x^2 - 2x - 3$ and $g(x) = -2x^2 + 12x - 18$.

Answer. To graph these functions, we graph the equations $y = f(x)$ and $y = g(x)$. So we can look at the graph of $y = x^2 - 2x - 3$ by plotting points and connecting the dots. We'll get the following.



Similarly, to graph g, we graph the equation $g(x) = -2x^2 + 12x - 18$.



We notice that the graphs in the previous example have a similar shape, though they have different orientation, width, and placement in the xy-plane. We give this shape a name, and analyze some of its qualities.

Definition. The graph of any quadratic function is called a *parabola*. Suppose a parabola is the graph of $f(x) = ax^2 + bx + c$. Then:

- If $a > 0$, the parabola opens upward. There is a point at the bottom of the parabola where the direction changes.
- If $a < 0$, the parabola opens downward. There is a point at the top of the parabola where the direction changes.
- In both cases, the point where the direction changes is called the *vertex*.
- The parabola is symmetric about the vertical line that passes through the vertex.
- As $|a|$ increases, the parabola gets wider. As $|a|$ decreases, the parabola gets narrower.

If we have an accurate graph, we can often estimate the coordinates of the vertex well. In the previous example, we see that the vertex of $f(x) = x^2 - 2x - 3$ seems to be the point $(1, -4)$, and that the vertex of $g(x) = -2x^2 + 12x - 18$ seems to be the point $(3, 0)$.
But we want to be able to find the vertex exactly, and without having to graph. Let's think about how. We know from Chapter 2 that the x-intercepts of the graph of $y = ax^2 + bx + c$ can be found using the quadratic formula:

$$x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Then by the symmetry of the parabola, we know the vertex must have x-coordinate exactly in between these intercepts! So the x-coordinate of the vertex is $x = \dfrac{-b}{2a}$. To find the y-coordinate, we can just plug this value of x into the function.

In order words, we have that the function $f(x) = ax^2 + bx + c$ has vertex at the point (h, k), where

$$h = \frac{-b}{2a} \text{ and } k = f(h).$$

Example 2. Find the vertex of the function $f(x) = 3x^2 - 12x + 4$.

Answer. Here, $a = 3$, $b = -12$, and $c = 4$. Therefore, $\dfrac{-b}{2a} = \dfrac{-(-12)}{2(3)} = \dfrac{12}{6} = 2$. So the x-coordinate of the vertex is 2.

To find the y-coordinate of the vertex, we plug $x = 2$ into the function and find the output. But $f(2) = 3(2)^2 - 12(2) + 4 = 12 - 24 + 4 = -8$.

Thus, the vertex is the point $(2, -8)$.

Using the completing the square trick that we used to find the quadratic formula in the first place, we can also write any quadratic function in a different form that immediately tells us the vertex.

Definition. Suppose $f(x) = ax^2+bx+c$ is a quadratic function. Then the *standard form* or *vertex form* of f is $f(x) = a(x-h)^2+k$, where (h,k) is the vertex of f. That is,

$$h = \frac{-b}{2a} \text{ and } k = f(h).$$

Note: The value of a is the same in both form of f. In other words, the leading coefficient of f will always be out front.

Example. We saw in Example 2 that the function $f(x) = 3x^2-12x+4$ has vertex at the point $(2,-8)$. This tells us that the vertex form of f is $f(x) = 3(x-2)^2-8$.

We can check this by expanding out the vertex form.

$$\begin{aligned} f(x) &= 3(x-2)^2-8 \\ &= 3(x^2-4x+4)-8 \\ &= 3x^2-12x+12-8 \\ &= 3x^2-12x+4 \end{aligned}$$

So we see that both forms give the same quadratic function.

The vertex form can also help us find the rule of a quadratic function if we know some information about its graph.

Example 3. Suppose the graph of a quadratic function has vertex $(-2,4)$ and passes through the point $(2,12)$. Find the formula for f.

Answer. We'll start by looking for the vertex form of f, $f(x) = a(x-h)^2+k$. So we need to find out the values of a, h, and k. But h and k are given by the coordinates of the vertex: $h=-2$ and $k=4$. Thus, $f(x) = a(x+2)^2+4$. So all we need to do is find a.

To get a, we'll use the information that the graph passes through the point $(2,12)$. If this point is on the graph, that tells us that $f(2) = 12$. We can figure out an expression for $f(2)$ in terms of a using our rule.

$$\begin{aligned} f(2) &= a(2+2)^2+4 \\ &= a(4)^2+4 \\ &= 16a+4. \end{aligned}$$

So to find a, we can set this equal to 12. That is, we solve the equation $16a+4=12$. Solving for a, we find that $a = \dfrac{1}{2}$.

Therefore, $f(x) = \dfrac{1}{2}(x+2)^2+4$. This is a good enough answer for us, but we could also expand this out if we wanted to, and we would get that $f(x) = \frac{1}{2}x^2+2x+6$.

Applications.
Again, consider $f(x) = ax^2 + bx + c$. We note that if $a > 0$, then the vertex is the lowest point on the parabola. That means that the y-coordinate of the vertex is the minimum possible output of the function. Similarly, if $a < 0$, then the vertex is the highest point on the parabola, so the y-coordinate of the vertex is the maximum possible output of the function. This is useful in applications when we want to maximize or minimize something.

Example 4. Suppose you shoot a bottle rocket into the air and start timing it with a stopwatch. Its height in feet can be thought of as a function of time. That is, suppose the height h of the rocket after t seconds is given by $h(t) = -16t^2 + 260t$.

(a) What is the maximum height reached by the rocket?
(b) How long does it take before the rocket hits the ground again?

Answer. (a) We know the maximum value of the function is the y-coordinate of the vertex. But since $a = -16$ and $b = 260$, we know that the x-coordinate of the vertex is $\dfrac{-260}{2(-16)} = \dfrac{-260}{-32} = \dfrac{65}{8}$.

So to find the maximum height, we plug in $t = \dfrac{65}{8}$ as input and the output will be our answer.

$$
\begin{aligned}
h\left(\frac{65}{8}\right) &= -16\left(\frac{65}{8}\right)^2 + 260\left(\frac{65}{8}\right) \\
&= -16\left(\frac{4225}{64}\right) + \frac{260 \cdot 65}{8} \\
&= -\frac{4225}{4} + \frac{4225}{2} \\
&= \frac{4225}{4} \\
&= 1056.25.
\end{aligned}
$$

So the maximum height of the rocket is 1056.25 feet.

(b) Now we want to find the time t that it takes for the rocket to hit the ground again. But if the rocket is on the ground, it has height of 0 feet. This means that we want to solve the equation $h(t) = 0$ for t. That is, we want to solve $-16t^2 + 260t = 0$. We can use either factoring or quadratic formula to solve this. If we factor, we see that $-4t(4t - 65) = 0$, so either $-4t = 0$ or $4t - 65 = 0$.

If $-4t = 0$, then $t = 0$. This is not the answer, because this just tells us that the rocket was on the ground before we launched it. We already knew that!

If $4t - 65 = 0$, then $t = \dfrac{65}{4} = 16.25$. This is our answer. The rocket is in the air for 16.25 seconds.

In the previous example, we were given the formula for the rule of the function that gave the height. In many examples, we may not be given this information.

Let's return to an example involving revenue, cost, and profit, also taking into account some of the ideas of supply and demand.

Example 5. Consider the wallet company from Example 1 in the Section 3.3 notes. They still manufacture wallets with fixed costs of \$2700 and a marginal cost per wallet of \$12. But their wallets haven't sold very well at the price of \$30 per wallet. There is only a demand for 125 wallets at that price. By researching their competitors, they are able to figure out a demand curve for their product, which they see is linear. In particular, they learn that for every dollar they lower the price, they can sell 25 more wallets. What price should they sell their wallets for to maximize their profit?

Answer. This is different from our other revenue and cost problems, because we don't know the price at which the item is going to be sold. So let's make that a variable. Here, I'll let p represent the sale price of a wallet. Our usual expressions for revenue, cost, and profit all depend on the quantity of wallets produced and sold, but here, that also depends on p. So the first step is to figure out how many wallets are in demand at price p. Let's make this a function, which we'll call N.

We know that when $p = 30$, the demand is 125 wallets, so $N(30) = 125$. If the price is lowered to $p = 29$, then the demand increases by 25, so $N(29) = 25 + 125 = 150$. Note that $30 - p$ represents the total price drop from \$30, and for each of those dollars dropped in price, N increases by 25. So we can come up with the function $N(p) = 25(30 - p) + 125$. If we want, we can expand this out and simplify, which gives us $N(p) = 875 - 25p$.

The next step is to use N to get an expression for the revenue in terms of p. We know that Revenue = (Price per item) × (Number of items sold). Therefore, we have that $R = p \times N(p)$. By simplifying, we get that

$$R(p) = p(875 - 25p) = 875p - 25p^2.$$

So now, revenue is a quadratic function of the price!
Similarly, we had that Cost = Fixed Cost + (Marginal cost per item) × (Number of items produced). The marginal cost per wallet is still \$12. We plug in $N(p)$ for the number of items produced, and we get

$$C(p) = 2700 + 12(875 - 25p) = 13200 - 300p.$$

Profit is still given by subtracting the costs from the revenue, so we have that

$$\begin{aligned} P(p) &= R(p) - C(p) \\ &= (875p - 25p^2) - (13200 - 300p) \\ &= 875p - 25p^2 - 13200 + 300p \\ &= -25p^2 + 1175p - 13200. \end{aligned}$$

So, profit is also a quadratic function, and the leading coefficient is negative, which means that the vertex is a maximum. So to find the price p that maximizes the profit, we just need to find the x-coordinate of the vertex for this quadratic function.

But then $p = \dfrac{-1175}{2(-25)} = 23.5$. Thus, the company should sell their wallets for \$23.50 each in order to maximize their profit.

SUMMARY:

- The graph of a quadratic function is a parabola.
- The vertex of the parabola given by $f(x) = ax^2 + bx + c$ is the point on the graph with x-coordinate of $\frac{-b}{2a}$.
- The y-coordinate of the vertex is always either the maximum or minimum value of the function, depending on the sign of the leading coefficient.