Section 4.1: Exponential Functions

In section 3.4, we studied quadratic functions, so we now understand a function like $f(x) = x^2$. Note that this function involves exponentiation, with the input x being the base and the number 2 being the exponent. Now, we're going to flip the roles, and use a number as the base and include the input x in the exponent.

Definition. An *exponential function* is any function where the input variable is included in the exponent.

We start with a particular simple form of an exponential function.

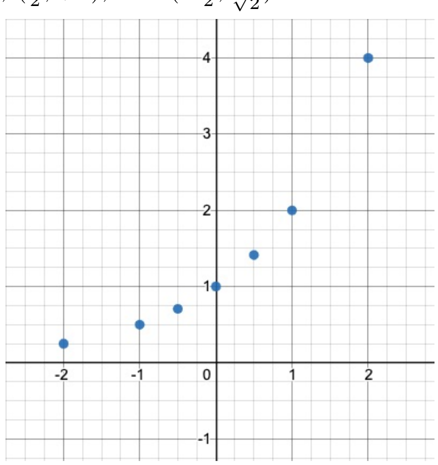
Definition. The function $f(x) = a^x$ is an exponential function, where a is a number. We call a the *base* of the exponential function, and we require that $a > 0$. (Often, we'll also assume that $a \neq 1$ as well.)

Example 1. Consider the exponential function $f(x) = 2^x$. Compute $f(1)$, $f(2)$, $f(0)$, $f(-1)$, $f(-2)$, $f\left(\frac{1}{2}\right)$, and $f\left(-\frac{1}{2}\right)$.

Answer. This might be a good time to review negative exponents, fractional exponents, and raising a number to the power of 0. So if you're having trouble seeing how we get these answers, go back to Section 1.5 to refresh your memory.

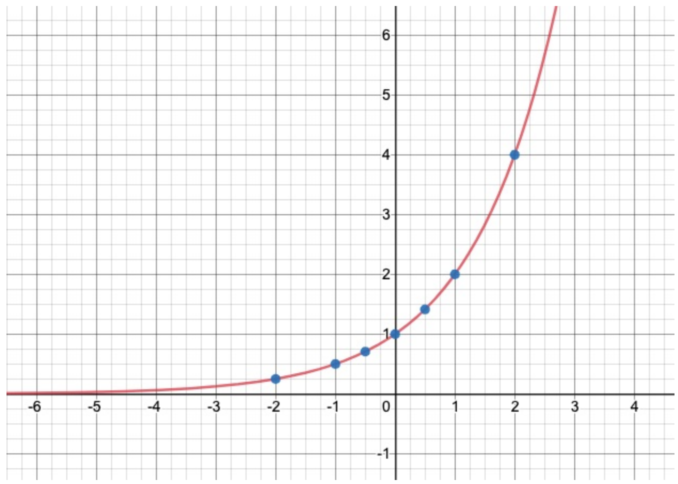
- $f(1) = 2^1 = 2$
- $f(2) = 2^2 = 4$
- $f(0) = 2^0 = 1$
- $f(-1) = 2^{-1} = \dfrac{1}{2}$
- $f(-2) = 2^{-2} = \dfrac{1}{2^2} = \dfrac{1}{4}$
- $f\left(\dfrac{1}{2}\right) = 2^{1/2} = \sqrt{2}$
- $f\left(-\dfrac{1}{2}\right) = 2^{-1/2} = \dfrac{1}{\sqrt{2}}$

Let's use those function values to sketch the graph of the function $f(x) = 2^x$. We start by plotting the points that are given by these function values, $(1, 2)$, $(2, 4)$, $(0, 1)$, $(-1, \frac{1}{2})$, $(-2, \frac{1}{4})$, $(\frac{1}{2}, \sqrt{2})$, and $(-\frac{1}{2}, \frac{1}{\sqrt{2}})$.



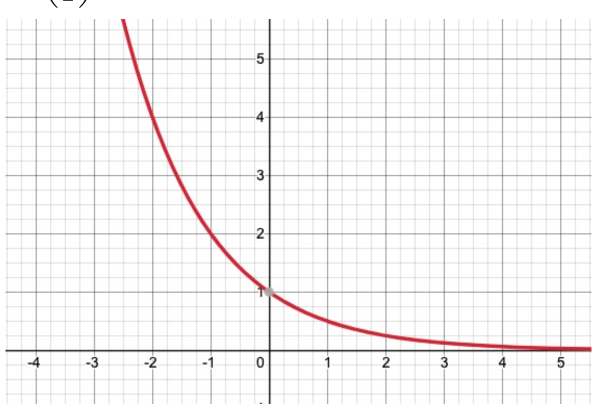
What happens on the left and on the right? Well, on the right, it's not too hard to see that the values of $f(x)$ will get bigger and bigger as the values of x increase. On the left, it looks like the values of $f(x)$ are getting smaller. But will they ever cross the x-axis? We have to ask what happens when we plug in very negative values of x.

Well, we can try it! We can note that $f(-1000) = \frac{1}{2^{1000}}$, which is very small, but still positive. Similarly, $f(-1000000)$ is even smaller, but is still positive. So we see that no matter what the input is, the output ALWAYS must be positive. So the graph can NEVER cross the x-axis. We can use these facts, together with the points we've plotted, to sketch the graph of f.



Perhaps the first thing we notice about this graph is how quickly the function values change. Every time we move 1 unit to the right along the x-axis, the function values double! This rapid speed growth is called *exponential growth.*

What happens to the graph if the base changes? Well, if the base a is bigger than 1, than the graph of $f(x) = a^x$ will still show this exponential growth, with the growth factor of a. That is, when we move 1 unit to the right, the function values multiply by a. If $0 < a < 1$, then we get something different. To see this, let's graph $f(x) = \left(\frac{1}{2}\right)^x$.



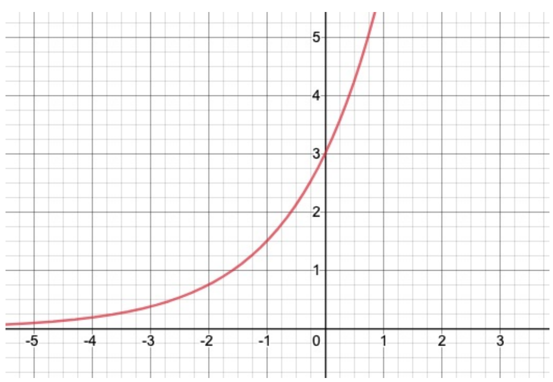
Then we get a sort of mirror image of the graph of 2^x. Now, the function values get smaller as the values of x increase. This is called *exponential decay*.

In both of these cases, by looking at the graph, we notice that the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$. Note that 0 is not included in the range because there are no x-intercepts. We also notice in both of these cases, there is a side of the graph that gets closer and closer to the x-axis without ever reaching it. This means that the x-axis is a *horizontal asymptote* of the graph. In other words, a horizontal asymptote of a graph is a horizontal line that the function values approach as the input values x either move to the left towards $-\infty$ or move to the right towards ∞.

We can also use these exponential expressions with the variable in the exponent as parts of more complicated functions. For example, we could multiple the exponential by a number.

Example 2. Graph the function $f(x) = 3 \cdot 2^x$.

Answer. We have to be careful to use the order of operations correctly when we read this. The order of operations says that exponents come before multiplication. So by adding in parentheses, we make sure that we read this as $f(x) = (3) \times (2^x)$. This is NOT the same as 6^x! Then $f(0) = 3 \cdot (2^0) = 3 \cdot 1 = 3$, $f(2) = 3 \cdot (2^2) = 3 \cdot 4 = 12$, etc. We get the following graph.

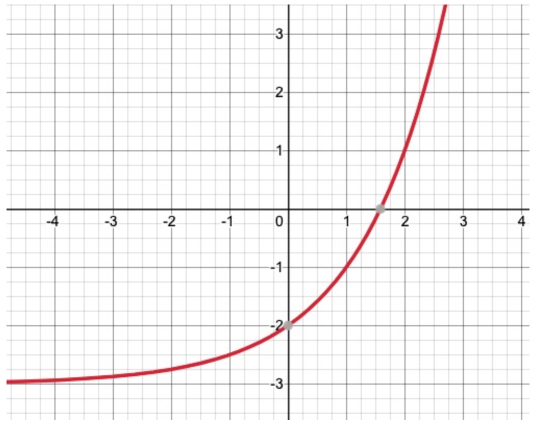


Note that this has the same domain, range, and horizontal asymptote as 2^x, but this graph is just a tiny bit steeper.

We could also add or subtract a number from an exponential expression.

Example 3. Graph the function $f(x) = 2^x - 3$, and determine the domain, range, and horizontal asymptote.

Answer. Now, $f(0) = 2^0 - 3 = 1 - 3 = -2$, $f(1) = 2^1 - 3 = -1$, etc. We get the following graph.



The domain is still $(-\infty, \infty)$, but now, since the graph has been shifted down, we have a different range. The range of f here is $(-3, \infty)$, and the horizontal asymptote is now the horizontal line $y = -3$. Although this graph DOES cross the x-axis, it never crosses the line $y = -3$.

Now that we can define and graph exponential functions with any base we want, we might ask the question, which base is best? Or which base is most useful?

Well, 10 is a very useful base, because it's the core of how we write numbers. That is, the digits of a number represent powers of 10. For example, can write $783 = 7 \cdot 10^2 + 8 \cdot 10^1 + 3 \cdot 10^0$.
Another very important base in modern times is 2, largely because of computers. Computers use a binary system, which is similar to our number system except using base 2. This is why the number of bytes in a KB is not 1000, but instead is $1024 = 2^{10}$!
But both mathematically and in many examples in nature, the most important base is a weird number between 2 and 3 that we call e. The number e is irrational, so it has an infinite decimal expansion that never repeats, but we can approximate e by $e \approx 2.718281828459$. If we were still having in-person meetings, I would probably spend 10 or 20 minutes telling you about the history of e, because it's my favorite number and has a good story behind it. If anyone wants me to write it out and share it with you, I will!

SUMMARY:

- Exponential functions are functions in which the input variable appears in the exponent of an exponential expression.

- The simplest exponential functions are of the form $f(x) = a^x$, where $a > 0$ is called the base.

- $f(x) = a^x$ has domain of $(-\infty, \infty)$, range of $(0, \infty)$, and has a horizontal asymptote at the x-axis.

- Changing the base of an exponential function changes the rate at which the function values get bigger or smaller.