Section 4.2: Applications of Exponential Functions

There won't be many new definitions in this section. Instead, the point here is to illustrate how versatile exponential functions are at explaining many phenomena we see in all aspects of life. In science, nature, business, medicine, and almost every imaginable profession, there are quantities that can be modeled with exponential functions. So in this section, we'll go through a few examples.

Example 1. (Population Growth)
A country keeps track of its population P in millions of people at time t, where t is measured in years since 1950. They have found that $P(t) = 35.3(1.0094^t)$. Using this model, what was the population in 1950? What was the population in 2000? What does the model predict that the population will be in 2025?

Answer. Since t measures the years since 1950, in order to find the population in the year 1950, we use $t = 0$. Then $P(0) = 35.3(1.0094^0) = 35.3(1) = 35.3$. Therefore, the population was 35.3 million people.

To find the population in 2000, we plug in $t = 50$, since 2000 was 50 years after 1950. Then $P(50) = 35.3(1.0094^{50})$. Using a calculator, we find that $1.0094^{50} \approx 1.5965$. Therefore, $P(50) = 35.3(1.5965) = 56.35645$. So, the population in 2000 was 56.35645 million people.

Finally, to find what the population will be in 2025, we plug in $t = 75$. Then $P(75) = 35.3(1.0094^{75}) = 71.207$. Thus, the population in 2025 is predicted to be 71.207 million people.

Populations of humans and animals often follow an exponential pattern when growth is unchecked. That is, the population will grow exponentially as long as there are sufficient resources. However, there is always a limit to the available resources. Imagine a species of squirrels in a particular forest. There is a maximum amount of food that the trees in the forest can produce for the squirrels, so the squirrel population is unable to sustainably grow past the amount that can be fed by the food supply. This is the idea behind *logistic growth*.

Example 2. (Logistic Growth)
The population p of fish in a certain lake after t months is given by the function $p(t) = \dfrac{20000}{1 + 24(2^{-.36t})}$. What was the population at the beginning of the period? What was the population after 2 years? What was the population after 4 years?

Answer. At the beginning of the period, we use $t = 0$. Then $p(0) = \dfrac{20000}{1 + 24(2^0)} = \dfrac{20000}{1 + 24} = 800$. So there were 800 fish in the lake initially.

After 2 years, a total of 24 months have passed, so we plug in $t = 24$. Then

$$p(24) = \frac{20000}{1 + 24(2^{-.36 \cdot 24})} = \frac{20000}{1 + 24(.0025)} = \frac{20000}{1 + .06016} = 18865.077.$$

So there are roughly 18865 fish in the lake after 2 years.

After 4 years, a total of 48 months have passed, so we plug in $t = 48$. Then

$$p(48) = \frac{20000}{1 + 24(2^{-.36 \cdot 48})} = \frac{20000}{1 + 24(.0000062835)} = \frac{20000}{1 + .0001508} = 19996.98.$$

So there are roughly 19997 fish in the lake after 4 years.

We note that although the population grew a lot in the first 2 years, it grew less in the next 2 years when resources became scarcer.

While these populations have shown exponential or logistic growth, other quantities decay exponentially. That is, the more of the population that is present, the more quickly it dies off. A classic example of this is radioactive elements.

Example 3. (Radioactive Decay)
A certain radioactive isotope has a half-life of 850 years. This means that no matter how much of the isotope there initially is, half of the initial amount will be remaining after 850 years. If a sample initially contains 1400 mg, how many mg will remain after 500 years?

Answer. We have to use the given information to express the amount A of the isotope remaining after t years. Our function should be in the form $A(t) = y_0 b^t$. We need to find the values of y_0 and b.

Since there is initially 1400 mg, that tells us that $A(0) = 1400$. Therefore, $y_0 b^0 = 1400$, so $y_0 = 1400$.

Next, we know that half of the sample is remaining after 850 years. Therefore, $A(850) = 700$. So, $1400b^{850} = 700$. Solving this, we get

$$b^{850} = \frac{1}{2}, \text{ so } b = \sqrt[850]{\frac{1}{2}} = \left(\frac{1}{2}\right)^{1/850}.$$

Thus, we can write our function as $A(t) = 1400\left(\frac{1}{2}\right)^{t/850}$. To solve the problem, we just have to plug in $t = 500$. We find that

$$A(500) = 1400\left(\frac{1}{2}\right)^{500/850} = 931.218.$$

So 931.218 mg of the isotope remain.

It turns out that temperature also changes according to an exponential function. The larger the difference is between the heat of an object and the ambient temperature, the faster the object will change temperature to match its environment. The general scientific principle that explains this is Newton's law of heating and cooling.

Example 4. A cup of coffee is brewed at a temperature of 170° F. In a 70° F room, it is found that the coffee cools according to Newton's law of cooling, so that its temperature F after t hours is given by $F(t) = 70 + 100\left(\frac{1}{64}\right)^t$. Find the temperature of the coffee after 20 minutes.

Answer. Since t is given in hours, we must convert 20 minutes to hours. So we use $t = \frac{1}{3}$. Thus,

$$F\left(\frac{1}{3}\right) = 70 + 100\left(\frac{1}{64}\right)^{1/3} = 95.$$

So the coffee is 95° F after 20 minutes.

Compound Interest.

The last major application that we'll focus on in these notes is not officially covered in the textbook until Sections 5.1 and 5.2, but we'll include it here. That application is the important financial topic of computing interest.

You might have encountered interest in a positive way, when your bank has paid you interest on funds in your savings account. You also might have encountered interest in a negative way, when you have had to pay extra interest on a loan or debt. How does this work?

Let's say you put your money into a bank account or savings account that gives interest. Such an account will have an interest rate, r, where $0 < r < 1$. This means that every so often, the bank will take the amount of money in your account, A, multiply it by r, and add that product Ar to account your. So your new balance becomes $A + Ar$, which we'll write as $A(1 + r)$. This process is called *compounding*. If the bank compounds the interest again, then it uses the new balance in the account to calculate it. So after the second compoundment, the account will have $A(1 + r) + A(1 + r)r$, which is $A(1 + r)(1 + r)$, or $A(1 + r)^2$. We can follow this pattern: if we initially have A dollars in the account with interest rate r, then the amount of money in the account after we compound N times is $A(1 + r)^N$.

How often is interest compounded? This will depend on the account, but it could be yearly, , twice a year, four times a year, monthly, or even more often perhaps. Let's use n to denote the number of times per year that interest is compounded, and we'll let t denote the number of years that have passed. That means that after t years, there will have been a total of nt compoundments.

Traditionally, r represents the annual interest rate, which means once a year. To compound interest more frequently, we want to use a prorated interest rate instead of the interest rate for the entire year. So for each of the n compoundments in a year, we'll use an interest rate of $\frac{r}{n}$.

Putting it all together, we get a formula for compound interest! That is, we let P be the principle, which is the initial amount of money in the account. We let r be the annual interest rate, and let n be the number of compoundments per year. Then the amount A of money in the account after t years is given by the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Example 5. You put \$1000 into an account with a .06 annual interest rate, compounded monthly. How much money is in the account after 3 years?

Answer. We use the formula. Since interest is compounded monthly, that means that $n = 12$ since there are 12 months in a year. Using $t = 3$, we get

$$A(3) = 1000\left(1 + \frac{.06}{12}\right)^{12\cdot 3} = 1196.68.$$

So the account has \$1196.68 after 3 years.

Next, one can ask the question of what happens to the formula as the number of compoundments per year increases more and more. If we compound daily, we use $n = 365$. If we compound every minute, we use $n = 525600$. What if we compound every second? Or every millisecond? Or even more?

In other words, we want to let n increase towards infinity. We call this *continuously compounded interest*. It turns out that doing this will lead to a result that involves our good friend, the number e. That is, if we put a principle P into an account with annual interest rate r that compounds continuously, the amount of money in the account after t years is given by

$$A(t) = Pe^{rt}.$$

Example 6. Suppose you put \$1000 into an account with a 4% annual interest rate, compounded continuously. How much money is in the account after 3 years?

Answer. Since the interest rate is 4%, that means that $r = .04$. Therefore, we get that

$$A(3) = 1000e^{.04\cdot 3} = 1127.50$$

Therefore, the balance in the account after 3 years is \$1127.50.

What if we asked the question: how long does it take before the balance in the account is \$1200? That means we would want to find the value of t such that $A(t) = 1200$. In particular, we would want to solve the equation $1000e^{.04t} = 1200$. How do we solve this for t? We need some way of undoing exponentiation. This will be the topic of the next section.

SUMMARY:

- The half-life of a radioactive isotope is defined to be the time it takes for half of the sample to decay.

- The amount of money after t years in an account with principle P with annual interest rate r that is compounded n times a year is given by $A(t) = P(1 + \frac{r}{n})^{nt}$.

- The amount of money after t years in an account with principle P with annual interest r that is compounded continuously is given by $A(t) = Pe^{rt}$.