Homework 10, due Friday, December 5

Please turn in well-written solutions for the following:

(1) The Fibonacci sequence is defined by $f_0 = 0$, $f_1 = 1$, and for any $n \geq 2$, we define f_n recursively by $f_n = f_{n-1} + f_{n-2}$. We use this to define a new sequence by $r_n = \dfrac{f_{n+1}}{f_n}$.

   (a) Use some simple algebra to show that $r_{n+1} = 1 + \dfrac{1}{r_n}$ for every n.

   (b) Assume that (r_n) is a convergent sequence. (This is true, but we may not have time to get through all the steps needed to show it.) In this case, use (a) to show that $\lim\limits_{n\to\infty} r_n = \dfrac{\sqrt{5}+1}{2}$, the golden ratio.

(2) Prove that $\limsup |s_n| = 0$ if and only if $\lim s_n = 0$.

(3) For each of the following sequences, compute $\limsup s_n$ and $\liminf s_n$.

   (a) $s_n = \frac{n-1}{n+1} \cos\left(\frac{n\pi}{3}\right)$

   (b) $s_n = \sqrt[n]{1 + 2^{n(-1)^n}}$

(4) Let (s_n) be a sequence of nonnegative numbers. For each n, we define the arithmetic average of the first n terms, $A_n = \frac{1}{n}(s_1 + s_2 + s_3 + \cdots + s_n)$.

   (a) For any M and N with $M > N$, show that

   $$\sup\{A_n : n > M\} \leq \frac{1}{M}(s_1 + \cdots + s_N) + \sup\{s_n : n > N\}.$$

   (b) Use part (a) to show that

   $$\liminf s_n \leq \liminf A_n \leq \limsup A_n \leq \limsup s_n.$$

   (c) Give an example of a sequence (s_n) such that $\lim A_n$ exists but $\lim s_n$ does not exist.