Homework 4, due Friday, September 19

Please turn in well-written solutions for the following:

(1) Let $D \subset \mathbb{R}$ and suppose $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are both continuous. Define $h : D \to \mathbb{R}$ by $h(x) = \max\{f(x), g(x)\}$. Prove that h is continuous on D.

(2) Consider the function

$$f(x) = \begin{cases} |x| & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

(a) Prove that f is continuous at 0.

(b) Prove that if $a \neq 0$, then f is not continuous at a.

(3) The Boundedness Theorem states that if $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then f is bounded on $[a, b]$. However, the fact that the interval is closed is a crucial part of this.
Give an example of a function f that is continuous on an open interval (a, b) such that f is unbounded on (a, b).

(4) The Extreme Value Principle states in part that if $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then f attains its maximum value. However, the fact that the interval is closed is once again a crucial part of this, even if the function is bounded.
Give an example of a function f that is continuous and bounded on an open interval (a, b) such that $\max\{f(x) : x \in (a, b)\}$ does not exist.