Homework 7, due Friday, October 24

Please turn in well-written solutions for the following:

(1) Use an ε-based argument to show that $f(x) = \dfrac{1}{x^2}$ is integrable on $[1, 2]$. (In other words, you may not simply cite the fact that f is monotone or continuous to conclude integrability.)

(2) Suppose that $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are both bounded, integrable functions. Prove that the product $h = fg$ is integrable on $[a, b]$.

(3) It is true that if a bounded function has a countable set of discontinuities, then it is integrable, but that proof is beyond the scope of this class. In these problems, we prove integrability for all functions with finitely many discontinuities, and for one function with countably infinite discontinuities.

  (a) Suppose that $f : [a, b] \to \mathbb{R}$ is bounded on all of $[a, b]$ and is continuous everywhere except at a single point, $x_0 \in [a, b]$. Prove that f is integrable on $[a, b]$.

  (b) Now suppose that $f : [a, b] \to \mathbb{R}$ is bounded on $[a, b]$ and is continuous at all but a finite amount of points, $\{x_1, x_2, x_3, \dots, x_n\} \subset [a, b]$. Prove that f is integrable on $[a, b]$.

  (c) Finally, we'll define a function that is not continuous on an infinite set of points. For each $k \in \mathbb{N}$, let $x_k = \dfrac{k}{k+1}$. Then $0 < x_k < 1$ for each k, and moreover, we have that

$$0 < x_1 < x_2 < x_3 < \cdots < x_k < x_{k+1} < \cdots < 1,$$

and $\sup\{x_k \mid k \in \mathbb{N}\} = 1$. Define a function $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, x_1) \cup [x_2, x_3) \cup [x_4, x_5) \cup \cdots \\ 0 & \text{if } x = 1 \text{ or } x \in [x_1, x_2) \cup [x_3, x_4) \cup \cdots \end{cases}$$

Prove that f is integrable on $[0, 1]$.

(4) (GRE Problem) A real-valued function f defined on $\mathbb{R}$ has the following property.
For every positive number ϵ, there exists a positive number δ such that

$$|f(x) - f(1)| \geq \epsilon \text{ whenever } |x - 1| \geq \delta.$$

This property is <u>equivalent</u> to which of the following statements about f?

(A) f is continuous at $x = 1$.
(B) f is discontinuous at $x = 1$.
(C) f is unbounded.
(D) $\lim_{|x| \to \infty} |f(x)| = \infty$.
(E) $\int_0^\infty |f(x)|\, dx = \infty$.