Homework 8, due Friday, October 31

Please turn in well-written solutions for the following:

(1) Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and $f(x) \geq 0$ for all $x \in [a, b]$. Prove that if $\int_a^b f(x)\, dx = 0$, then $f(x) = 0$ for all $x \in [a, b]$. (Hint: Use contrapositive or contradiction and use a lemma about continuous functions that are positive at a point from earlier in the course.)

(2) Recall that we defined $L(x) = \int_1^x \frac{1}{t}\, dt$, we showed that L is a bijection from $(0, \infty)$ onto $\mathbb{R}$, and we defined E to be the inverse function of L. We showed that $E(0) = 1$, and that $E(x + y) = E(x)E(y)$ for all $x, y \in \mathbb{R}$. Then we used this to define the operation of raising a positive real number x to the power of any other real number s by

$$x^s = E(sL(x)).$$

(a) Under this definition and the properties of E listed above, show that for any s, $t \in \mathbb{R}$, we have that $x^{s+t} = x^s x^t$.

(b) We also showed that $E'(x) = E(x)$ for any real number x. Let s be an arbitrary nonzero real number, and define $f : [0, \infty) \to \mathbb{R}$ by $f(x) = x^s$. Use the above definition of x^s based on E, the differentiation property of E, and the chain rule to show that $f'(x) = sx^{s-1}$.

(3) Note that for any real numbers α and β, we have that

$$\int_a^b (\alpha f(x) + \beta g(x))^2 \, dx \geq 0.$$

(a) Use this to prove that for any α, $\beta \in \mathbb{R}$ and for any integrable functions f and g on $[a, b]$, we have that

$$2\alpha\beta \int_a^b f(x)g(x)\, dx \leq \alpha^2 \int_a^b (f(x))^2\, dx + \beta^2 \int_a^b (g(x))^2\, dx.$$

(b) By picking the values of α and β in a clever way, use part (a) to show that

$$\left(\int_a^b f(x)g(x)\, dx \right)^2 \leq \left(\int_a^b (f(x))^2 \, dx \right) \left(\int_a^b (g(x))^2 \, dx \right).$$

This result is an integral form of the Cauchy-Schwarz Inequality.

(c) Use part (b) to show that

$$\left(\int_a^b (f(x) + g(x))^2 \, dx \right)^{1/2} \leq \left(\int_a^b (f(x))^2 \, dx \right)^{1/2} + \left(\int_a^b (g(x))^2 \, dx \right)^{1/2}.$$

This result is a special case of Minkowski's Inequality.