Homework 2, due Friday, January 30

Please turn in well-written solutions for the following problems:

(1) Let (X, d) be any metric space.

(a) Prove that for any $x_0 \in X$, the set $\{x_0\}$ is closed.

(b) Prove that any finite set is closed.

(2) (1.4.5 in Tao) Let (X, d) be a metric space, $E \subseteq X$. Recall that we defined x to be an *accumulation point* or *adherent point* of a set E if there exists a sequence $(x_n)_{n=1}^{\infty}$ in E such that $x_n \to x$. Recall that we defined x to be a *cluster point* of a sequence $(x_n)_{n=0}^{\infty}$ if for all $\varepsilon > 0$ and for all $N \geq 1$, there exists $n \geq N$ such that $d(x_n, x) < \varepsilon$.

(a) Suppose that $(x_n)_{n=1}^{\infty}$ is a sequence in a metric space (X, d). Prove that if x is a cluster point of the sequence, then x is an adherent point of the set $\{x_n : n \geq 1\}$.

(b) Prove that the converse is false. That is, prove that there exists a sequence $(x_n)_{n=1}^{\infty}$ in some metric space (X, d) such that x is an adherent point of the set $\{x_n : n \geq 1\}$, but such that x is not a cluster point of the sequence.

(3) (1.4.6 in Tao) Let (X, d) be a metric space, and $(x_n)_{n=1}^{\infty}$ be a sequence in X. Prove that if L_1 and L_2 are both cluster points of (x_n) and $L_1 \neq L_2$, then (x_n) is not Cauchy.

In addition, I suggest that you study these problems from Tao:

- Section 1.4, problems 1.4.3, 1.4.2, 1.4.8