Homework 3, due Friday, February 13

Please turn in well-written solutions for the following problems:

(1) (1.5.12 in Tao) Let (X, d_{disc}) be a metric space with the discrete metric.

   (a) Prove that (X, d_{disc}) is always complete.

   (b) Under what conditions on the set X do we have that (X, d_{disc}) is compact? When is (X, d_{disc}) not compact?

(2) (1.5.14 in Tao) Let (X, d) be a metric space, let $E \subseteq X$ be compact and non-empty, and let $x_0 \in X$. Show that there exists a point $x \in E$ such that

$$d(x_0, x) = \inf\{d(x_0, y) : y \in E\},$$

i.e., x is the closest point in E to x_0. (Hint: Let $R = \inf\{d(x_0, y) : y \in E\}$, and construct a sequence (x_n) in E with $d(x_0, x_n) \leq R + \frac{1}{n}$. Then use compactness.)

(3) (1.5.15 in Tao plus more) Let (X, d) be a compact metric space. Suppose that $(K_\alpha)_{\alpha \in I}$ is a collection of closed sets in X such that any finite subcollection of these sets has non-empty intersection. That is, for any finite set $F \subseteq I$, we have that $\bigcap_{\alpha \in F} K_\alpha \neq \emptyset$. (This property is called the *finite intersection property.*)

   (a) Prove that if $(K_\alpha)_{\alpha \in I}$ has the finite intersection property, then the grand intersection $\bigcap_{\alpha \in I} K_\alpha$ is non-empty.

   (b) Show by counterexample that part (a) is false if X is not compact.

   (c) Prove, however, that if every collection of closed subsets of X that has the finite intersection property also has a non-empty grand intersection, then X must be compact. (Hint: Given an open covering $(V_\alpha)_{\alpha \in I}$, consider the collection of closed sets given by $(V_\alpha^c)_{\alpha \in I}$.)

In addition, I suggest that you study these problems from Tao:

- Section 1.5, problems 1.5.2, 1.5.8, 1.5.9, 1.5.10 (not easy!), 1.5.13