Homework 4, due Friday, February 20

Please turn in well-written solutions for the following problems:

(1) Let $(X, d_X), (Y, d_Y)$ be metric spaces. We define a function $f : X \to Y$ to be *open* if for each open set $V \subset X$, the image $f(V)$ is open in Y.

  (a) If f is open, is it continuous? Prove or find a counterexample.

  (b) If f is continuous, is it open? Prove or find a counterexample.

(2) (2.3.5 in Tao) Prove that multiplication in $\mathbb{R}$ is not uniformly continuous. That is, consider $\mathbb{R}^2$ with the l^2 metric and $\mathbb{R}$ with the standard metric, and consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = xy$. Show that f is not uniformly continuous on $\mathbb{R}$.

(3) (2.4.8 in Tao plus more) Let (X, d) be a metric space, and let $E \subseteq X$.

  (a) Show that if E is connected, then $\overline{E}$ is connected.

  (b) Is the converse true? That is, if E is disconnected, does it follow that $\overline{E}$ is disconnected?

  (c) You saw in Theorem 2.4.6 that if $f : X \to Y$ is a continuous function and $E \subseteq Y$ is connected, then $f(E)$ is connected. That is, f takes connected sets to connected sets. Is it also true that f takes disconnected sets to disconnected sets? Prove or find a counterexample.

In addition, I suggest that you study these problems from Tao:

- Section 2.1, problems 2.1.4, 2.1.5
- Section 2.2, problems 2.2.3, 2.2.10
- Section 2.3, problems 2.3.3, 2.3.4
- Section 2.4, problems 2.4.1, 2.4.2, 2.4.6, 2.4.7