

HOMEWORK 6, DUE FRIDAY, MARCH 13

Please turn in well-written solutions for the following problems:

- (1) For each positive integer n , let $f_n : [-\pi, \pi] \rightarrow \mathbb{R}$ by $f_n(x) = \cos(x + n)$. Show that $(f_n)_{n=1}^{\infty}$ has a subsequence that converges uniformly to a continuous function on $[-\pi, \pi]$.
- (2) In class, we used the word equicontinuity of a collection F of functions from X to Y to refer to *uniform equicontinuity*, which says that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $f \in F$ and for all $x_1, x_2 \in X$, whenever $d(x_1, x_2) < \delta$ we have that $d(f(x_1), f(x_2)) < \varepsilon$.

However, for a point $x_0 \in X$, we also give the following definition. We say that F is *pointwise equicontinuous at x_0* if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $f \in F$ and for all $x \in X$, whenever $d(x_0, x) < \delta$ we have that $d(f(x_0), f(x)) < \varepsilon$. Then we say that F is *pointwise equicontinuous on X* if F is pointwise equicontinuous at x_0 for every $x_0 \in X$.

- (a) Suppose that X is a compact metric space and F is a collection of functions from X to Y . Prove that if F is pointwise equicontinuous on X , then F is uniformly equicontinuous.
 - (b) The statement in (a) is false if X is not assumed to be compact. Show this by finding a counterexample on \mathbb{R} . That is, find a collection of functions F from \mathbb{R} to \mathbb{R} that is pointwise equicontinuous on \mathbb{R} but is not uniformly equicontinuous. (Hint: Your collection F doesn't need to have many functions in it.)
 - (c) Is the converse of (a) true? That is, if every pointwise equicontinuous sequence of functions is automatically uniformly equicontinuous, does that imply that X is compact? Prove or give a counterexample.
- (3) Recall that we defined $\text{Lip}_K(X \rightarrow \mathbb{R})$ to be the set of all functions f in $B(X \rightarrow \mathbb{R})$ such that $|f(x_1) - f(x_2)| \leq K d_X(x_1, x_2)$ for all $x_1, x_2 \in X$, where $K > 0$. Suppose that X is compact.

- (a) Prove that $\text{Lip}_K(X \rightarrow \mathbb{R})$ is equicontinuous.
- (b) Prove that $\text{Lip}_K(X \rightarrow \mathbb{R})$ is not uniformly bounded. In particular, find a sequence $(f_n)_{n=1}^{\infty}$ of functions such that $f_n \in \text{Lip}_K$ for every n , but that $\lim_{n \rightarrow \infty} \sup\{f_n(x) : x \in X\} = \infty$.