Homework 9, due Monday, April 20

Please turn in well-written solutions for the following problems:

(1) (5.2.6 in Tao) Let $f \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$, and let $(f_n)_{n=1}^{\infty} \subset C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$.
   (a) Show that if $f_n \to f$ uniformly, then $f_n \to f$ in the L^2 metric.
   (b) Give an example where $f_n \to f$ in the L^2 metric, but $f_n \not\to f$ uniformly. (Hint: Take $f = 0$ and modify an example from a previous homework problem.)
   (c) Give an example where $f_n \to f$ in the L^2 metric, but $f_n \not\to f$ pointwise.
   (d) Give an example where $f_n \to f$ pointwise, but $f_n \not\to f$ in the L^2 metric. (Hint: Take $f = 0$ and again, consider a similar question from a previous homework.)

(2) (5.5.3 in Tao) Let $f, g \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$. Recall (or note for the first time) the definition of the *convolution* $f * g : \mathbb{R} \to \mathbb{C}$, by the formula

$$(f * g)(x) = \int_0^1 f(y)g(x - y)dy.$$

In this exercise, you will show that the Fourier transform turns convolution into multiplication.
   (a) If P is a trigonometric polynomial, prove that $\widehat{f * P}(n) = \widehat{f}(n)\widehat{P}(n)$.
   (b) Prove that $\widehat{f * g}(n) = \widehat{f}(n)\widehat{g}(n)$.

(3) (5.5.4 in Tao) Suppose that $f \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ is differentiable and that f' is continuous. Show that $f' \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ and that $\widehat{f'}(n) = 2\pi i n \widehat{f}(n)$ for any integer n.

(4) (5.5.5 in Tao) Let $f, g \in C(\mathbb{R}/Z, \mathbb{C})$. In this exercise, you will prove two more different related results that are also given the name *Parseval's Identity.*
   (a) Show that

$$\operatorname{Re}\left(\int_0^1 f(x)\overline{g(x)}dx\right) = \operatorname{Re}\left(\sum_{n \in \mathbb{Z}} \widehat{f}(n)\overline{\widehat{g}(n)}\right).$$

   (Hint: Consider applying Plancherel's theorem to $f + g$ and $f - g$.)
   (b) Conclude that

$$\int_0^1 f(x)\overline{g(x)}dx = \sum_{n \in \mathbb{Z}} \widehat{f}(n)\overline{\widehat{g}(n)}.$$

   (Hint: apply (a) with f replaced by if.)

In addition, I suggest that you study these problems from Tao:

- Section 4.7, problems 4.7.7, 4.7.10
- Section 5.1, problem 5.1.3
- Section 5.2, problems 5.2.1, 5.2.2, 5.2.3, 5.2.4, 5.2.5
- Section 5.3, problem 5.3.5
- Section 5.5, problems 5.5.1, 5.5.2