

FINAL EXAM MATERIAL AND EXPECTATIONS

For the final exam, you should be able to do the following things:

Chapter 1.

- Add, subtract, multiply, and divide complex numbers
- Find complex conjugate, modulus, and argument of a complex number
- Convert between $x + iy$ form and polar form $re^{i\theta}$
- Multiply, divide, and take powers and roots of complex numbers in polar form
- Solve basic equations in \mathbb{C}
- Sketch regions in the complex plane
- Determine whether regions are open, closed, connected, bounded, etc.

Chapter 3.

- Compute values of functions evaluated at complex numbers, including polynomials, rational functions, complex exponential, and branches of complex logarithms and radicals
- Write general form of a multivalued function such as logarithms or radicals in terms of an arbitrary integer n
- Determine largest domain of definition of a function

Chapter 2.

- Compute limits in \mathbb{C} , and determine when limits do not exist
- Compute limits involving the point at ∞
- Use derivative formulas for complex functions that mirror the formulas of the real analogues
- Use the Cauchy-Riemann equations to determine whether a complex function given in terms of real and imaginary parts is analytic at a point
- Use Cauchy-Riemann equations to find harmonic conjugates. That is, given $u(x, y)$ harmonic, find $v(x, y)$ harmonic such that $f(x, y) = u(x, y) + iv(x, y)$ is analytic

Chapter 4.

- Compute integrals of functions from \mathbb{R} to \mathbb{C}
- Write typical parametrizations of simple contours such as line segments or arcs of a circle
- Use parametrization to write an integral of a complex function over a contour in \mathbb{C} as an integral over a real interval:

$$\int_C f(z) dz = \int_a^b f(z(t))z'(t) dt$$

- Use the Cauchy-Goursat Theorem and related results:
 - If f has an antiderivative F on a domain containing the contour C from z_1 to z_2 , then $\int_C f(z) dz = F(z_2) - F(z_1)$
 - f is analytic everywhere in a simply connected domain D if and only if f has an antiderivative on D
 - If f is analytic on and inside a closed simple contour C , then $\int_C f(z) dz = 0$
- Use the Cauchy Integral Formula: If f is analytic on and inside a simple closed positive contour C around a point z_0 , then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz \text{ and for any } n \in \mathbb{N}, f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Chapter 5.

- Determine convergence/limit of sequences or series of complex numbers by comparing to the sequences or series of real and imaginary parts
- Use the geometric series and the Maclaurin series of e^z , $\sin(z)$, and $\cos(z)$:
 - $\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n$ for $|z| < 1$
 - $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ for any $z \in \mathbb{C}$
 - $\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$ for any $z \in \mathbb{C}$
 - $\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$ for any $z \in \mathbb{C}$
- Use the above Maclaurin series to write Laurent series of rational functions and functions involving exponentials and trig functions

Chapter 6.

- Find singularities of a function and determine whether they are removable, essential, poles, or non-isolated
- Compute the residue of a function at an essential singularity using Laurent series
- Compute the residue of a function at a pole of higher order by Laurent series or by writing $f(z) = \phi(z)(z - z_0)^{-m}$ and calculating

$$\text{Res}(f, z_0) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

- Compute the residue of a function at a simple pole by either of the above techniques, or using

$$\text{Res}\left(\frac{p(z)}{q(z)}, z_0\right) = \frac{p(z_0)}{q'(z_0)}$$

- Compute integrals using the Cauchy Residue Theorem: If f has only finitely many singularities interior to a positive simple closed contour C , then

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f, z_k)$$

Chapter 7.

- Use Cauchy Residue Theorem and semicircular contours to compute improper integrals of real-valued functions over \mathbb{R}
- Use Cauchy Residue Theorem, semicircular contours, and e^{ix} to compute improper integrals from Fourier analysis
- Use Cauchy Residue Theorem, indented semicircular contours, and complex functions with branch cuts to compute improper integrals of real-valued functions with logarithms or radicals
- Use Cauchy Residue Theorem and substitutions for $\sin(\theta)$ and $\cos(\theta)$ to compute integrals of functions involving \sin and \cos over $[0, 2\pi]$