Problem 1.8

Given: $m\ddot{x} + kx = 0$

$k = 4\ N/m$

$m = 1\ kg$

$X_0 = 1\ mm = 0.001\ m$

$V_0 = 0$

Find: a) $x(t)$

b) plot $x(t)$

Solution: $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4}{1}} = 2\ rad/s$

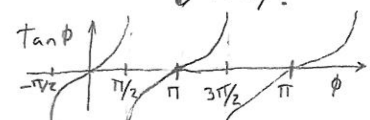
$f_n = \frac{\omega_n}{2\pi} = \frac{2}{2\pi} = 0.318\ Hz$ (cycles/sec)

$T = \frac{1}{f_n} = 3.14\ s$

$A = \frac{\sqrt{\omega_n^2 X_0^2 + V_0^2}}{\omega_n} = \frac{\sqrt{2^2 \times 0.001^2 + 0^2}}{2} = 0.001\ m = 1\ mm$

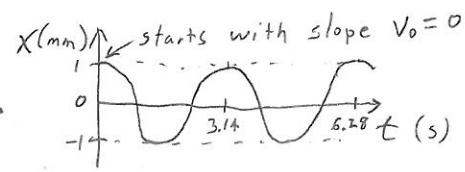
$\phi = \tan^{-1}\left(\frac{\omega_n X_0}{V_0}\right) = \tan^{-1}\left(\frac{2 \times 0.001}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$

↓ Why?

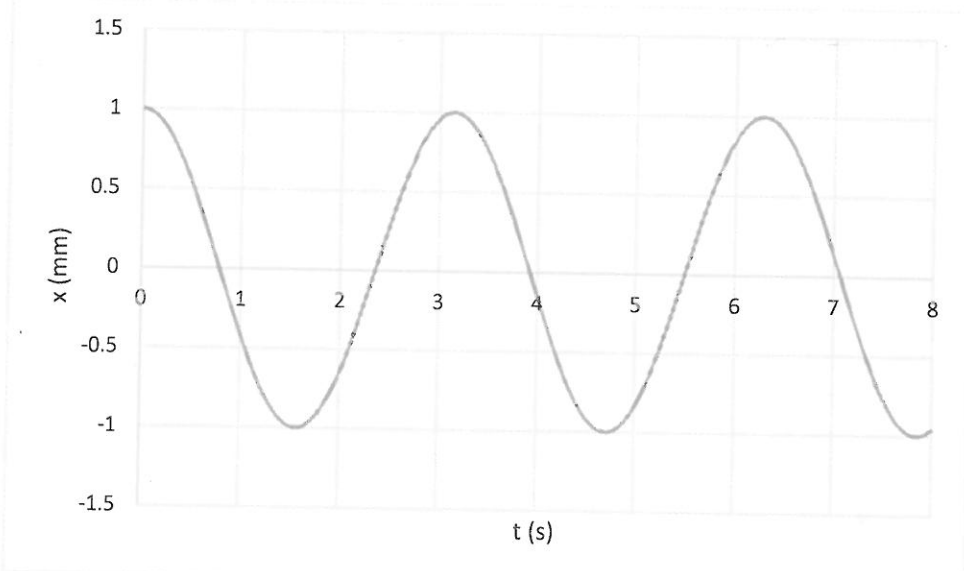


$X(t) = A\sin(\omega_n t + \phi)$

a) $\boxed{x(t) = 0.001\ \sin(2t + \pi/2)\ m}$ $\xrightarrow{\text{sketch}}$

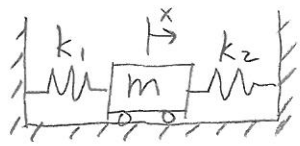


b)

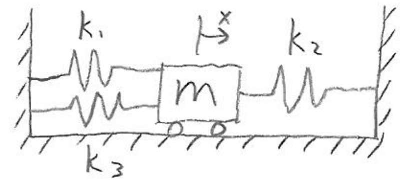


Problem 1.18

Given: a)

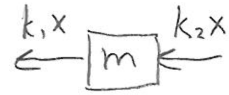


b)



Find: ω_n

Solution:



$$\overset{+}{\rightarrow}\sum F_x = m\ddot{x}$$

$$-k_1x - k_2x = m\ddot{x}$$

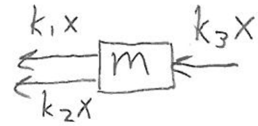
$$m\ddot{x} + (k_1+k_2)x = 0$$

$\downarrow$ compare with

$$m\ddot{x} + kx = 0 \rightarrow \omega_n = \sqrt{\frac{k}{m}}$$

$\downarrow$ therefore

$$\omega_n = \sqrt{\frac{k_1+k_2}{m}}$$



$$\overset{+}{\rightarrow}\sum F_x = m\ddot{x}$$

$$-k_1x - k_2x - k_3x = m\ddot{x}$$

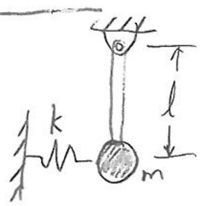
$$m\ddot{x} + (k_1+k_2+k_3)x = 0$$

$$\downarrow$$

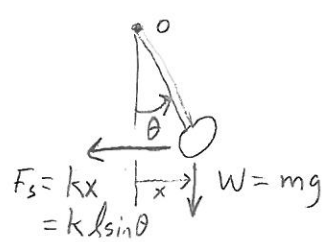
$$\omega_n = \sqrt{\frac{k_1+k_2+k_3}{m}}$$

Problem 1.21

Given:



$\longrightarrow$



$x = l\sin\theta$

Find: ω_n

Solution: $\overset{+}{\circlearrowleft}\sum M_o = I_o\ddot{\theta}$

$$-mg \times l\sin\theta - kl\sin\theta \times l\cos\theta = (ml^2)\ddot{\theta}$$

$$\ddot{\theta}ml^2 + (mgl + kl^2\cos\theta)\sin\theta = 0$$

$$\ddot{\theta} + \left(\frac{g}{l} + \frac{k}{m}\cos\theta\right)\sin\theta = 0$$

For small θ, $\sin\theta \approx \theta$, $\cos\theta \approx 1$, therefore

$$\ddot{\theta} + \left(\frac{g}{l} + \frac{k}{m}\right)\theta = 0 \xrightarrow{\text{compare with}} m\ddot{x} + kx = 0$$

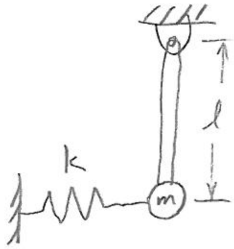
$$\downarrow$$

$$\boxed{\omega_n = \sqrt{\frac{g}{l} + \frac{k}{m}}} \xleftarrow{\text{therefore}} \omega_n = \sqrt{\frac{k}{m}}$$

Homework 1.11, 1.16, 1.22, 1.23

Problem 1.21 with Initial Conditions

Given:



$m = 0.1$ kg

$l = 150$ mm $= 0.15$ m

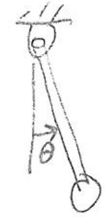
$k = 10$ N/m

$\theta_0 = 10° = 0.1745$ rad

$\dot{\theta}_0 = -128°/s = -2.234$ rad/s

$g = 9.81$ m/s²

- Assume small θ



Find: a) $\theta(t)$

b) Plot $\theta(t)$

Solution: From before $\omega_n = \sqrt{\frac{g}{l} + \frac{k}{m}} = \sqrt{\frac{9.81}{0.15} + \frac{10}{0.1}} = 12.86$ rad/s

$f_n = \frac{\omega_n}{2\pi} = \frac{12.86}{2\pi} = 2.05$ Hz

$T = \frac{1}{f_n} = 0.489$ s

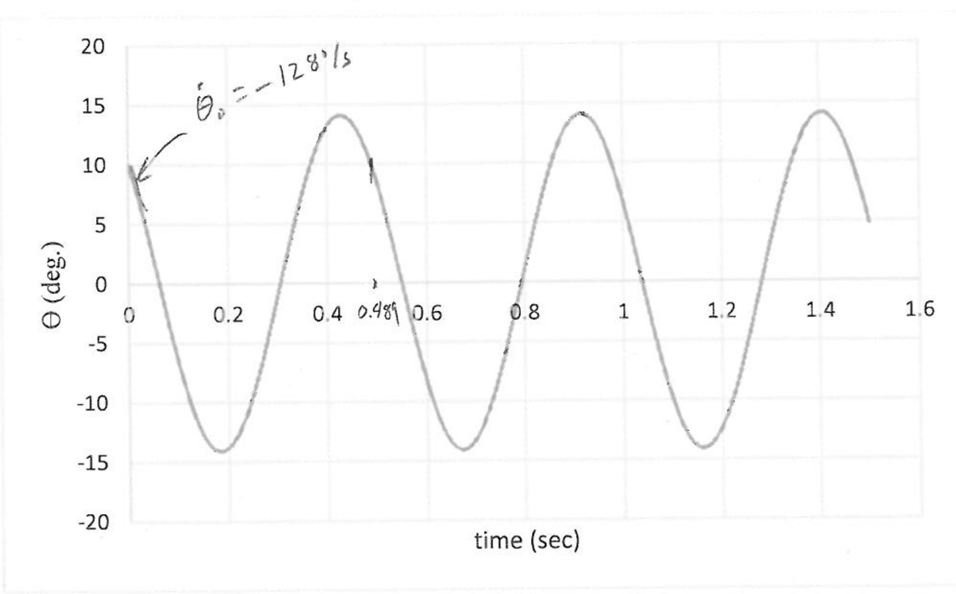
$$A = \frac{\sqrt{\omega_n^2 \theta_0^2 + \dot{\theta}_0^2}}{\omega_n} = \frac{\sqrt{12.86^2 \times 0.1745^2 + (-2.234)^2}}{12.86} = 0.246 \text{ rad} = 14.1°$$

$$\phi = \tan^{-1}\left(\frac{\omega_n \theta_0}{\dot{\theta}_0}\right) = \tan^{-1}\left(\frac{12.86 \times 0.1745}{-2.234}\right) = -45.1°$$

- But this is in quadrant II $\therefore \phi = -45.1 + 180 = 134.9° = 2.35$ rad

a) $\theta(t) = 0.246 \sin(12.86t + 2.35)$ rad

b)



Homework 1.19 (also, if $x_0 = 0$, what v_0 will give $A = 0.2$ m?)

1.20

1.24

Problem 1.13

Given: $m\ddot{x} + kx = 0$

$$x(t) = A_1 \sin \omega_n t + A_2 \cos \omega_n t$$

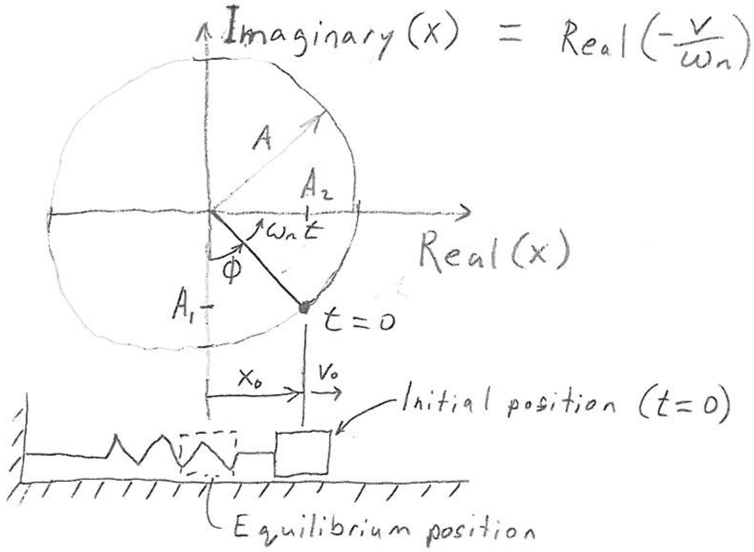
Find: A_1 & A_2 in terms of x_0 and v_0

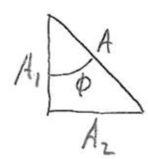
Solution: $\dot{x}(t) = A_1 \omega_n \cos \omega_n t - A_2 \omega_n \sin \omega_n t$

$$x_0 = x(0) = A_1 \cancel{\sin(0)}^{0} + A_2 \cancel{\cos(0)}^{1} =$$

$$v_0 = \dot{x}(0) = A_1 \omega_n \cancel{\cos(0)}^{1} - A_2 \omega_n \cancel{\sin(0)}^{0} = A_1 \omega_n$$

$$\therefore \; A_1 = \frac{v_0}{\omega_n} \; , \quad A_2 = x_0$$





$$x(t) = A \sin(\omega_n t + \phi)$$

or

$$x(t) = A_1 \sin(\omega_n t) + A_2 \cos(\omega_n t)$$

$$A = \sqrt{A_1^2 + A_2^2}$$

$$\phi = \tan^{-1}\left(\frac{A_2}{A_1}\right)$$

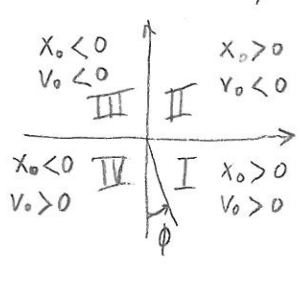
$$A_1 = A \cos \phi$$

$$A_2 = A \sin \phi$$

With imaginary component

$$x(t) = (A_1 \sin \omega_n t + A_2 \cos \omega_n t) + (-A_1 \cos \omega_n t + A_2 \sin \omega_n t)\, j$$

Be careful using $\phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right)$ or $\phi = \tan^{-1}\left(\frac{A_2}{A_1}\right)$ since it will only return ϕ values in quadrants I & IV.



- Add 180° to ϕ if in quadrants II or III.
- Alternatively, use ATAN2 $(x_0, \frac{v_0}{\omega_n})$ or rectangular to polar coordinate conversion.

Summary of Important Equations

$m\ddot{x} + kx = 0$ ← system differential equation

$x(t) = A\sin(\omega_n t + \phi)$

$\dot{x}(t) = \omega_n A\cos(\omega_n t + \phi)$

$\ddot{x}(t) = -\omega_n^2 A\sin(\omega_n t + \phi)$

$\omega_n = \sqrt{\frac{k}{m}}$

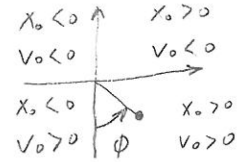
$x_0 = A\sin\phi$

$v_0 = \omega_n A\cos\phi$

or

$A = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n}$

$\phi = \tan^{-1}\frac{\omega_n x_0}{v_0}$ ← Remember to correct for quadrant



Natural frequency in Hz $f_n = \frac{\omega_n}{2\pi}$

Period $T = \frac{1}{f_n} = \frac{2\pi}{\omega_n}$

Harmonic motion amplitudes: disp.: A

vel.: $A_v = \omega_n A$

acc.: $A_a = \omega_n^2 A$

Converting between $x(t) = A\sin(\omega_n t + \phi)$ and $x(t) = A_1\sin\omega_n t + A_2\cos\omega_n t$

$A = \sqrt{A_1^2 + A_2^2}$

$\phi = \tan^{-1}\left(\frac{A_2}{A_1}\right)$

$A_1 = A\cos\phi$

$A_2 = A\sin\phi$