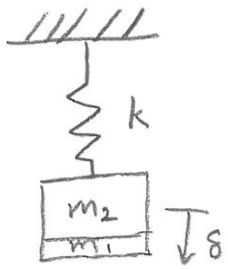
Chapter 1.6 Measuring m, k, and c

Do laser measurement first

I Measuring mass properties

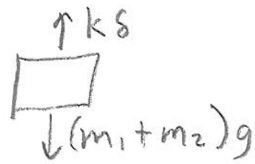
a) measuring mass

- static measurement



scale
k is known
m_1 is known
m_2 is unknown
δ can be measured (static deflection)

free body



some constants

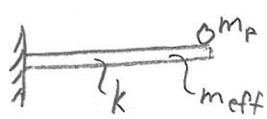
$$\boxed{m_2 = \frac{k\delta}{g} - m_1 = C_1\delta + C_2}$$

How are C_1 & C_2 (or k & m_1) determined in the first place?

1. By design (e.g. spring designed to give specific k (or C_1); pan designed for specific m_1 (or C_2)
2. By calibration (e.g., turn dial to set $\delta = 0$ when $m_2 = 0$)

→ Another solution is to use a balance!

- dynamic mass measurement
  - Example: Microcantilever particle sensor



k is known ← cantilever stiffness
m_{eff} is unknown ← cantilever effective mass
m_p is unknown ← particle mass
$m_p << m_{eff}$

ω_{n_1} is measured ← nat. freq. without m_p
$\Delta\omega_n$ is measured ← change in nat. freq. when particle becomes attached
$\Delta\omega_n << \omega_{n_1}$

$$\omega_{n_1} = \sqrt{\frac{k}{m_{eff}}} \qquad \therefore\ m_{eff} = \frac{k}{\omega_{n_1}^2}$$

$$\Delta\omega_{n_1} = \omega_{n_2} - \omega_{n_1} = \sqrt{\frac{k}{m_{eff} + m_p}} - \omega_{n_1}$$

$$= \sqrt{\frac{k}{\frac{k}{\omega_{n_1}^2} + m_p}} - \omega_{n_1}$$

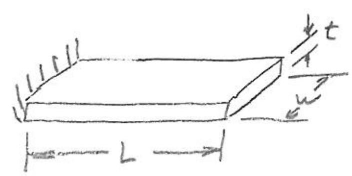
$$m_p = \frac{k}{(\Delta\omega_n + \omega_{n_1})^2} - \frac{k}{\omega_{n_1}^2}$$

$$= \frac{k\omega_{n_1}^2 - k(\Delta\omega_n + \omega_{n_1})^2}{(\Delta\omega_n + \omega_{n_1})^2\,\omega_{n_1}^2}$$

$$= \frac{\cancel{-k\Delta\omega_n^2} - 2k\Delta\omega_n\omega_{n_1}}{(\cancel{\Delta\omega_n +}\,\omega_{n_1})^2\,\omega_{n_1}^2} \qquad \text{- since } \Delta\omega_n \ll \omega_{n_1}$$

$$\boxed{m_p = \frac{-2k\Delta\omega_n}{\omega_{n_1}^3}}$$

Example: For a silicon cantilever with length $L = 50\,\mu m$, width $w = 10\,\mu m$, thickness $t = 1\,\mu m$, find the mass of the particle if $\omega_{n_1} = 419\,kHz$ and $\Delta\omega_n = -48\,Hz$.



Sol'n: For silicon $E = 1.5 \times 10^{11}\ N/m^2$

For cantilever $k = \dfrac{3EI}{L^3} = \dfrac{3E(\frac{1}{12}wt^3)}{L^3} = \dfrac{Ewt^3}{4L^3}$

$$= \frac{(1.5\times10^{11})(10\times10^{-6})(1\times10^{-6})^3}{4(50\times10^{-6})^3}$$

$$= 3\ N/m$$

$\omega_{n_1} = 419\ kHz = 2.63\times10^{6}\ rad/s$

$\Delta\omega_n = -48\ Hz = -302\ rad/s$

$$m_p = \frac{-2k\Delta\omega_n}{\omega_{n_1}^3} = \frac{-2(3)(-302)}{(2.63\times10^6)^3} = 9.96\times10^{-17}\ kg$$

$$\boxed{m_p = 1\times10^{-16}\ kg}$$

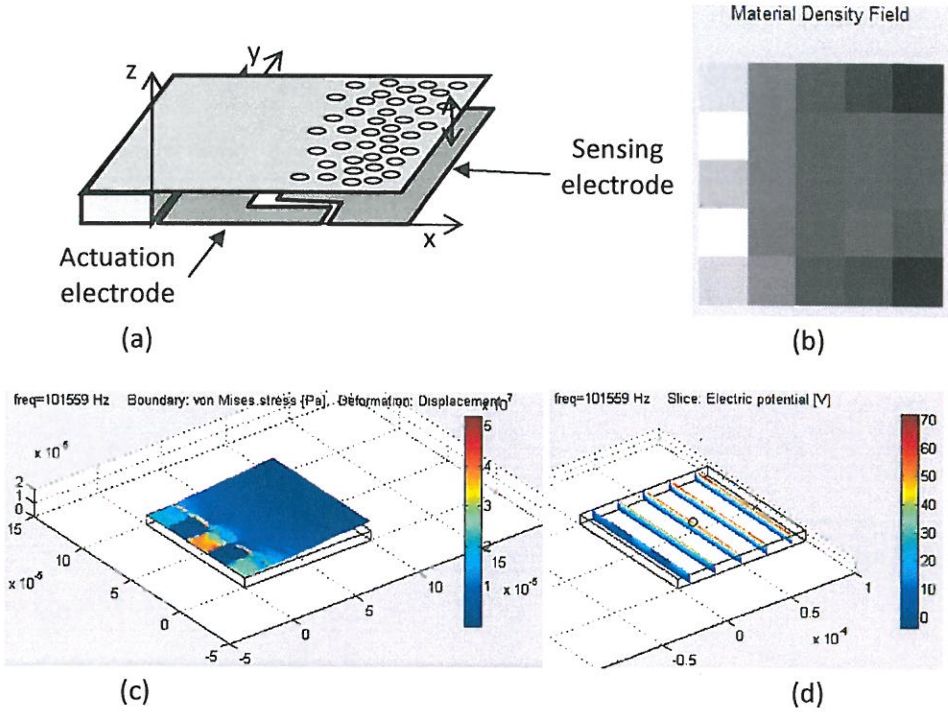


Figure 1. Micro-Electromechanical Sensor Design Optimization Problem; (a) Structure with micropores above the actuation and sensing electrodes, (b) optimal structure, (c) structural dynamics model, (d) electrostatic model.

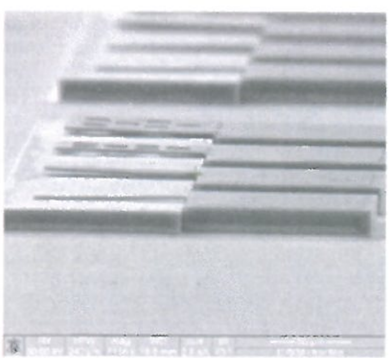


Figure 2. SEM perspective view of released fixed-free cantilevers

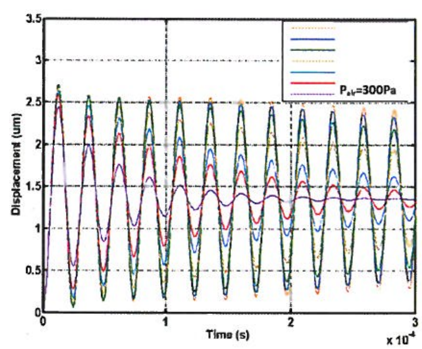


Figure 3. Step responses in different air pressures

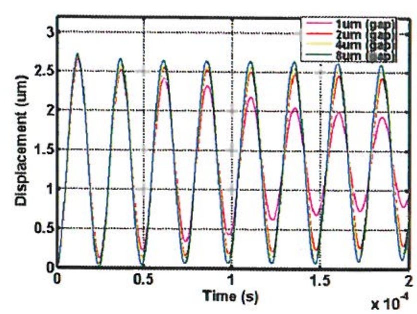
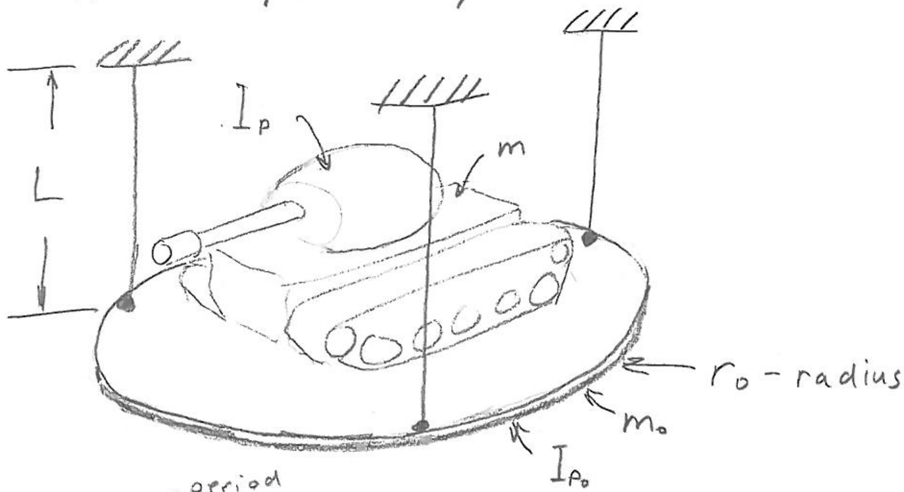


Figure 4. Step responses with different gap sizes

Note that these are only preliminary results.

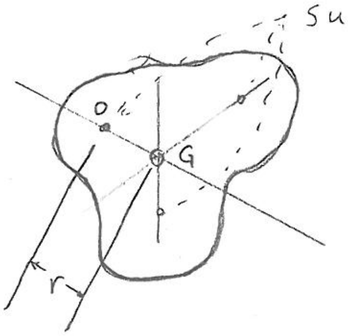
b) Measuring mass moment of inertia

E.g., Trifilar Suspension System



$$I_p = \frac{g \overset{\text{period}}{T^2} r_0^2 (m_0 + m)}{4\pi^2 L} - I_{P_0}$$

Another Example of Solving for mass moment of inertia



suspension points - The center of gravity (G) will always lie directly below this point when the object is suspended from this point

- Then suspend from one of the points (o) and measure natural frequency

$$I_{P_0}\ddot{\theta} + mgr\theta = 0$$

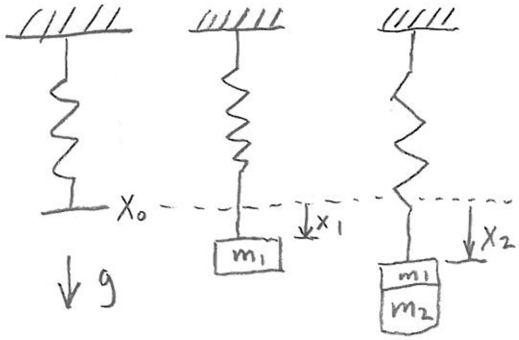
$$\omega_n = \sqrt{\frac{mgr}{I_{P_0}}} \rightarrow \boxed{I_{P_0} = \frac{mgr}{\omega_n^2}}$$

$$I_{P_0} = I_{P_G} + mr^2$$

$$\boxed{I_{P_G} = I_{P_0} - mr^2}$$

II Measuring Stiffness

- static measurement



ΔX is measured
g is known
Δm is known $\quad \Delta m = (m_1+m_2) - m_1 = m_2$

$$\Delta X = X_2 - X_1 = \frac{(m_1+m_2)g}{k} - \frac{m_1 g}{k} = \frac{m_2 g}{k}$$

$$\boxed{k = \frac{\Delta m\, g}{\Delta X}}$$

- dynamic measurement

- provide initial velocity or displacement and measure frequency of free vibration
  - m is known
  - ω_n is measured

$$\boxed{k = \omega_n^2 m}$$

- Can also measure modulus of elasticity (E)

Known: m, I, L
measured: T ← period of vibration

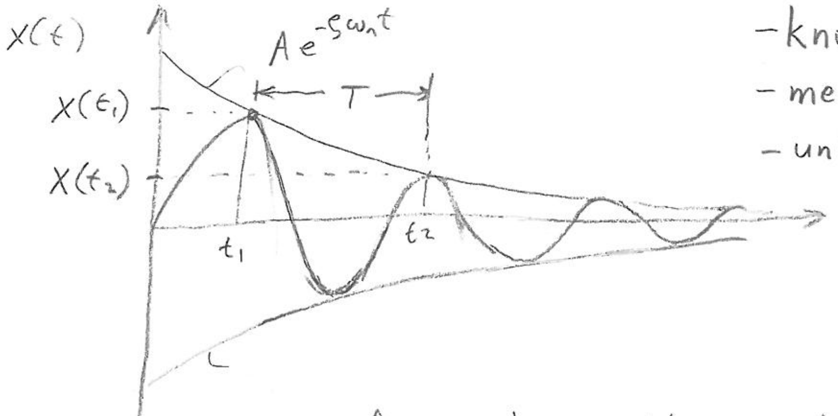
$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3EI}{mL^3}} = \frac{2\pi}{T}$$

$\therefore$

$$\boxed{E = \frac{4\pi^2 m L^3}{3T^2 I}}$$

III. Measuring Damping Coefficient

- has to be a dynamic measurement
- most difficult of the three to measure



- known: ω_n
- measured: $x(t_1), x(t_2)$
- unknown: ζ

Define logarithmic decrement

$$\boxed{\delta = \ln \frac{x(t)}{x(t+T)}}$$

$$\delta = \ln \frac{Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)}{Ae^{-\zeta\omega_n(t+T)}\sin(\omega_d(t+T)+\phi)}$$

$\omega_d T = 2\pi$, therefore

$$\delta = \ln \frac{e^{-\zeta\omega_n t}}{e^{-\zeta\omega_n(t+T)}}$$

$$= \ln e^{\cancel{-\zeta\omega_n t}-(-\zeta\omega_n(\cancel{t}+T))}$$

$$= \zeta\omega_n T$$

$T = \frac{2\pi}{\omega_d}$, therefore

$$\delta = \zeta\omega_n \frac{2\pi}{\omega_d} = \zeta\omega_n \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

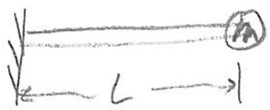
$$\boxed{\zeta = \frac{\delta}{\sqrt{4\pi^2+\delta^2}}}$$

For multiple periods use

$$\delta = \frac{1}{n}\ln\frac{x(t)}{x(t+nT)}$$ number of periods

Problem 1.97

Given: cantilever, composite material ($E = ?$)



$L = 1\,m$

$I = 10^{-9}\,m^4$

$m = 6\ kg$

oscillation period $T = 0.5\,s$

Find: E

Sol'n: For cantilever vibrating with mass at the end, neglecting weight of cantilever

$$k = \frac{3EI}{L^3} \qquad \omega_n = \sqrt{\frac{3EI}{mL^3}}$$

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{mL^3}{3EI}}$$

$$E = \frac{4\pi^2 m L^3}{3IT^2}$$

$$= \frac{4\pi^2(6)(1)^3}{3(10^{-9})(0.5)^2}$$

$$\boxed{E = 3.16 \times 10^{11}\,Pa}$$

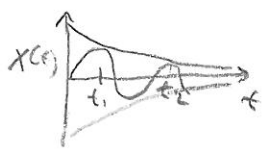
Problem 1.98

Given:

model as SDOF system

$m = 1000$ kg

$k = 400{,}000$ N/m



$x(t_1) = 0.02$ m

$x(t_2) = 0.0022$ m

Find: c

Sol'n: 1) $\delta = \ln \dfrac{x(t_1)}{x(t_2)} = \ln \dfrac{0.02}{0.0022} = 2.21$

2) $\zeta = \dfrac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \dfrac{2.21}{\sqrt{4\pi^2 + 2.21^2}} = 0.331$

3) $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{400{,}000}{1000}} = 20$ rad/s

4) $\zeta = \dfrac{c}{c_{cr}} = \dfrac{c}{2\sqrt{km}}$

$c = \zeta 2 \sqrt{km} = 0.331(2)\sqrt{400{,}000 \times 1000}$

$\boxed{c = 13260 \text{ N·s/m}}$

Homework 1.94, 1.95