Numerical Simulation of Time Response

Section 1.9

Introduction

- So far all of the vibration problems could be represented by linear differential equations of the form:
- Non-linear differential equations do not have such simple solutions:

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0$$

$$m\ddot{x} + k_1 x + k_2 x^3 = 0$$

- These problems can be solved using numerical methods.

The "Euler Method"

- Several numerical methods are available. We will use the "Euler Method."
- Assume:

$$\frac{dx}{dt} \approx$$

- If we use time steps:

$$\begin{aligned} t_0 &= 0 & x_0 &\Leftarrow \text{given} \\ t_1 &= \Delta t & x_1 &= ? \\ t_2 &= 2\Delta t & x_2 &= ? \\ t_3 &= 3\Delta t & x_3 &= ? \end{aligned}$$

The "Euler Method"

then: $\frac{dx}{dt} \approx \frac{x_{i+1} - x_i}{\Delta t}$

- As an example, to solve the D.E.: $\dot{x} = ax$

  then $\frac{x_{i+1} - x_i}{\Delta t} = ax_i$ ⇒ $x_{i+1} = x_i + \Delta t a x_i$

- All we **need** is the initial displacement $\boldsymbol{x_0}$ and to choose a good value for $\boldsymbol{\Delta t}$. Then we can compute an estimate of $x(t)$ for each time step.
- The **smaller** we set $\boldsymbol{\Delta t}$, the **better** the estimate will be.

2nd Order Differential Equations

- How about the D.E.: $m\ddot{x} + c\dot{x} + kx = 0$?
- Split it into two problems:

$$z_1(t) = x(t)$$
$$z_2(t) = \dot{x}(t)$$

- Taking derivatives with respect to time:

$$\dot{z}_1(t) = \dot{x}(t) = z_2$$
$$\dot{z}_2(t) = \ddot{x}(t) = -\frac{c}{m}\dot{x} - \frac{k}{m}x = -\frac{c}{m}z_2 - \frac{k}{m}z_1$$

2nd Order Differential Equations

- In matrix form:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

"State variables"

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z}$$

"State vector"

"State matrix"

- Use Euler equation to solve numerically:

$$\mathbf{z}_{i+1} = \mathbf{z}_i + \Delta t \mathbf{A} \mathbf{z}_i$$

2nd Order Differential Equations

- To use this method, we need to:
  1) Create the **A** matrix.
  2) Initialize $\mathbf{z}_0 = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}$
  3) Set Δt to a good value.
  4) Repeatedly compute $\mathbf{z}_{i+1} = \mathbf{z}_i + \Delta t \mathbf{A} \mathbf{z}_i$

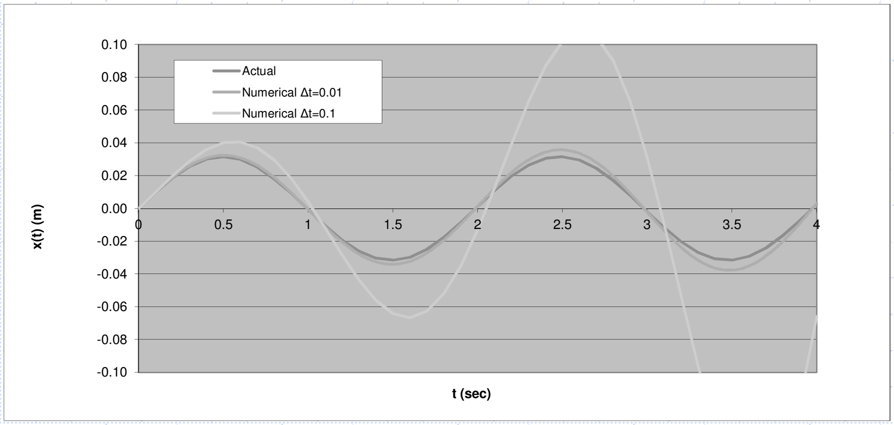
     for $i = 1 \text{ to } \dfrac{t_{final}}{\Delta t}$

Other Numerical Methods

- Other, more accurate methods are available, some that even calculate Δt automatically (E.g., Runge-Kutta method).
- These methods are available as functions in Matlab and MathCAD.

Accuracy of Numerical Simulation

m	0.1 kg	x0	0 m	ωn	3.16228 rad/s	A	0.031623 m
k	1 N/m	v0	0.1 m/s			Φ	0 rad

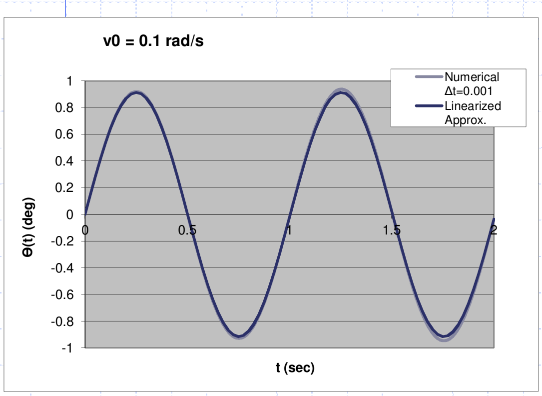


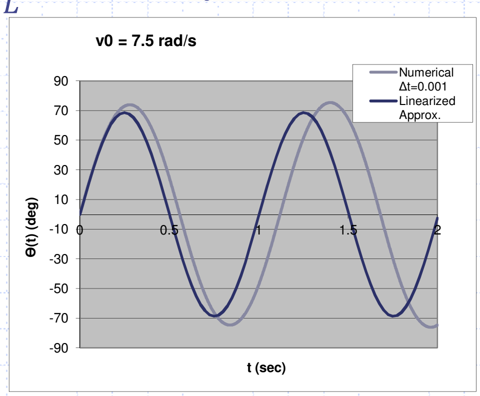
Smaller time steps generally yield more accurate solutions.

Linearization of Non-Linear System

Now that we know how to get an accurate simulation, let's compare the numerical ("actual") solution of $\ddot{\theta} + \frac{g}{L}\sin(\theta) = 0$ with the linearized solution $\ddot{\theta} + \frac{g}{L}\theta = 0$.

L	0.25 m	x0	0 rad
g	9.81 m/s^2		





In this case, the linearization under-predicts the amplitude and period.