Institute of Technology
West Virginia University

Harmonic Excitation of Undamped Systems

Section 2.1
Forced Vibration!

Motorcycle Engine Vibration Problem

- A motorcycle engine turns (and vibrates) at **5000 rpm** with a harmonic force of **20 N**.

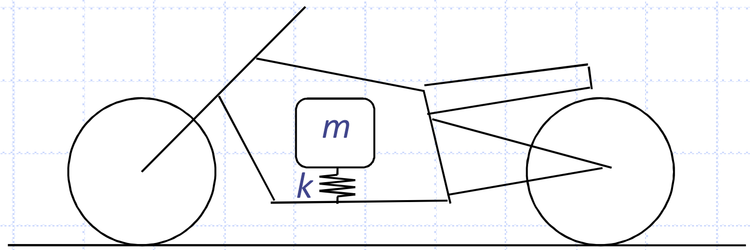
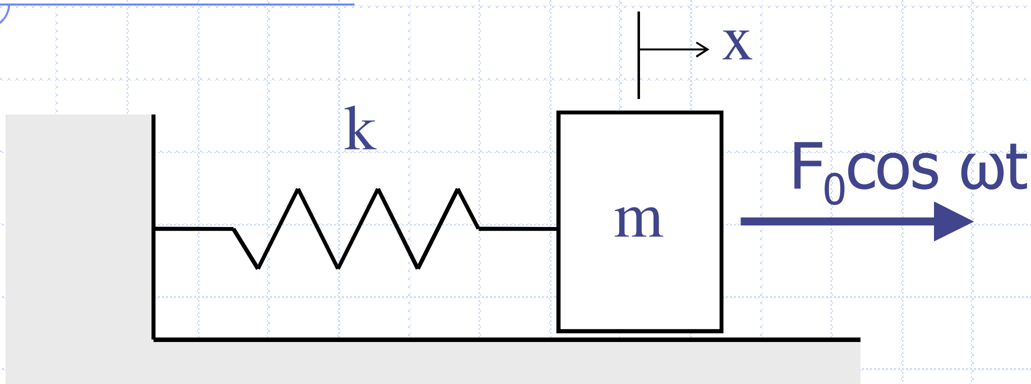


Figure P1. Motor vibrating in motorcycle

- What is the amplitude of the vibration with respect to the frame (assumed to be stationary) and phase of the response (with respect to the force), if:
  - the mass of the engine is **40 kg**,
  - the stiffness of the mounts is **40 kN/m**?

Undamped Spring-Mass System with Forced Vibration



$$\sum F_x = m\ddot{x}$$

$$-kx + F_0\cos(\omega t) = m\ddot{x}$$

$$\boxed{m\ddot{x} + kx = F_0\cos(\omega t)}$$

$$x(t) = x_h(t) + x_p(t)$$

Homogeneous problem	Particular problem
$m\ddot{x} + kx = 0$	$m\ddot{x} + kx = F_0\cos(\omega t)$
$x_h(t) = A\sin(\omega_n t + \phi)$	$x_p(t) = X\cos(\omega t)$

Solving for *X*

Differentiating $x_p(t)$ with respect to t:

$$\dot{x}_p(t) = -\omega X \sin(\omega t)$$

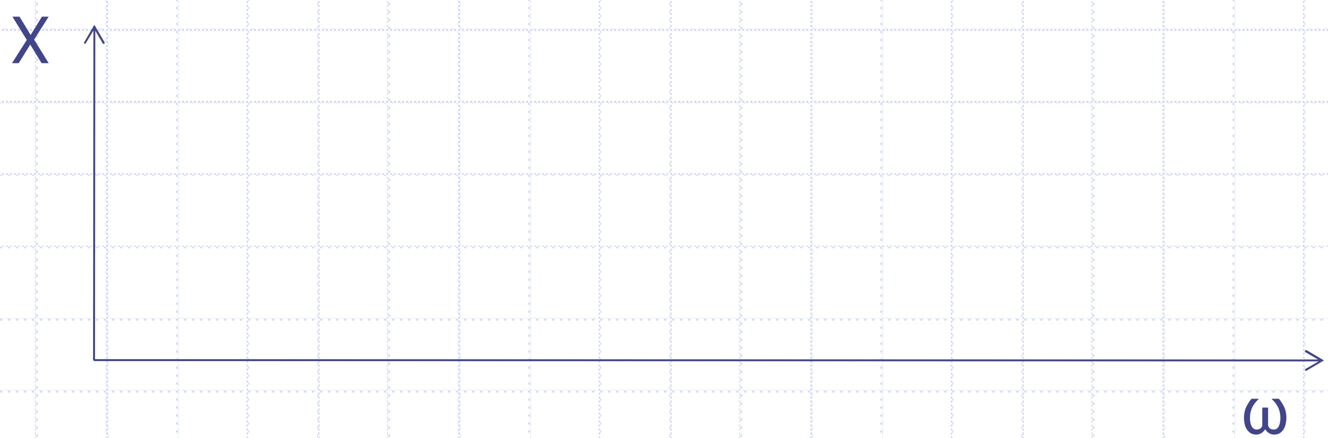
$$\ddot{x}_p(t) = -\omega^2 X \cos(\omega t)$$

Substituting back into D.E.:

$$-m\omega^2 X \cos(\omega t) + kX \cos(\omega t) = F_0 \cos(\omega t)$$

Solving for X:

$$\boxed{X = \frac{F_0}{k - m\omega^2}}$$



Solving for *A* and *Φ*

- The **particular solution** must also be accounted for when **solving for *A* and *Φ***. Therefore new equations are needed.
- To make things easier, we use A_1 and A_2 instead of *A* and *Φ* :

$$x(t) = A_1 \sin \omega_n t + A_2 \cos \omega_n t + \frac{F_0}{k - m\omega^2} \cos \omega t$$

$$x_0 = x(0) = A_2 + \frac{F_0}{k - m\omega^2}$$

$$v_0 = \dot{x}(0) = \omega_n A_1$$

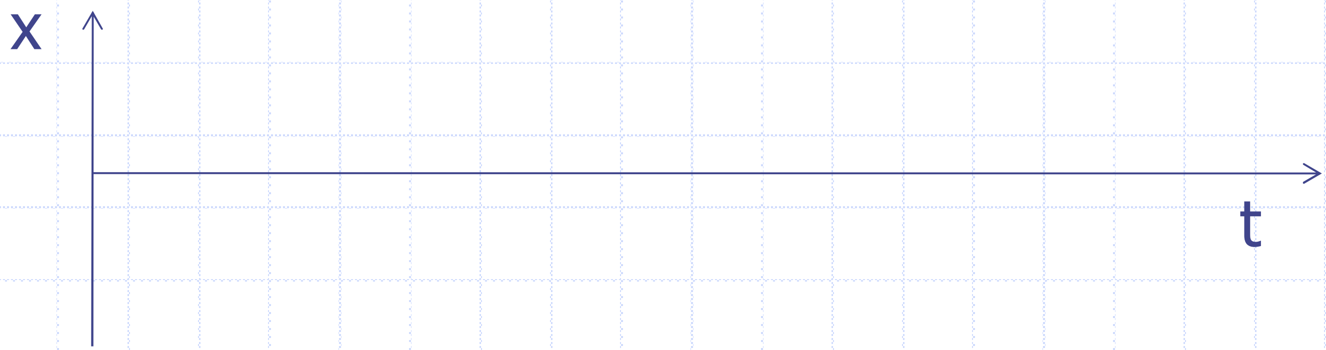
Solving for *A* and *Φ*

- Solving for A_1 and A_2 and inserting into $x(t) = x_h(t) + x_p(t)$ yields:

$$x(t) = \frac{v_0}{\omega_n}\sin\omega_n t + \left(x_0 - \frac{F_0}{k - m\omega^2}\right)\cos\omega_n t + \frac{F_0}{k - m\omega^2}\cos\omega t$$

Explaining "Beat" Phenomena

- What "beat" looks like:



- Start with overall solution to D.E.:

$$x(t) = \frac{v_0}{\omega_n}\sin\omega_n t + \left(x_0 - \frac{F_0}{k - m\omega^2}\right)\cos\omega_n t + \frac{F_0}{k - m\omega^2}\cos\omega t$$

- Let $x_0 = 0$ and $v_0 = 0$:

$$x(t) = \frac{F_0}{k - m\omega^2}\left(\cos\omega t - \cos\omega_n t\right)$$

Explaining "Beat" Phenomena

- Since: $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$$x(t) = \frac{2F_0}{k - m\omega^2}\sin\left(\frac{\omega_n - \omega}{2}t\right)\sin\left(\frac{\omega_n + \omega}{2}t\right)$$

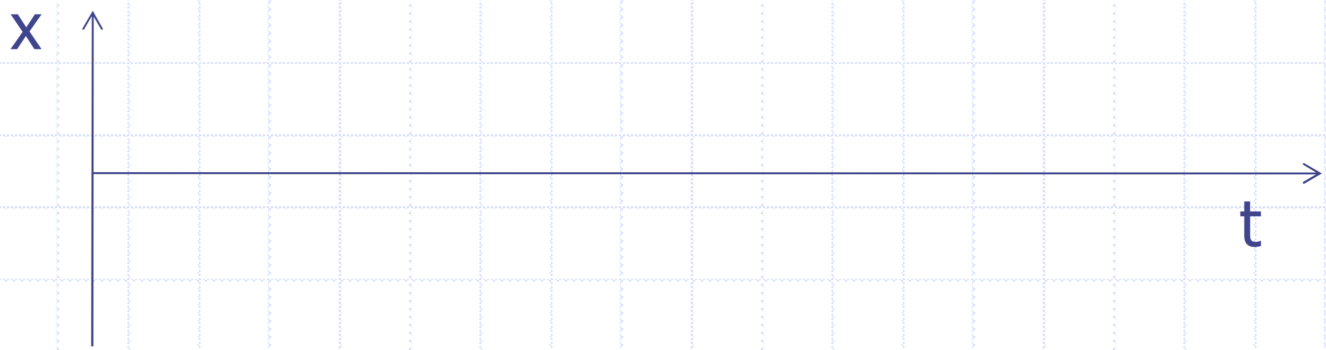
- If $\omega_n - \omega$ is small (i.e., ω is close to ω_n), then 2 components are observed:
  - High frequency oscillation component that produces vibration at a frequency of $\boxed{\omega_{osc} = (\omega_n + \omega)/2} \approx \omega \approx \omega_n$.

$$x(t) = \frac{2F_0}{k - m\omega^2}\underbrace{\sin\left(\frac{\omega_n - \omega}{2}t\right)}\overbrace{\sin\left(\frac{\omega_n + \omega}{2}t\right)}$$

  - Low frequency component that produces beat at a frequency of $\boxed{\omega_{beat} = |\omega_n - \omega|}$.

Explaining "Resonance" Phenomena

- What "resonance" looks like:



- If $\omega = \omega_n$, the particular solution to $m\ddot{x} + kx = F_0 \cos(\omega_n t)$ can not be $x_p(t) = X \cos(\omega_n t)$ because that is also a solution to the homogeneous problem.

Explaining "Resonance" Phenomena

- We must therefore use a solution of the form:

$$x_p(t) = tX\sin(\omega_n t) \Rightarrow x_p(t) = \frac{F_0}{2m\omega_n} t\sin(\omega_n t)$$

- Therefore

$$x(t) = A_1 \sin\omega_n t + A_2 \cos\omega_n t + \frac{F_0}{2m\omega_n} t\sin\omega_n t$$

$$= \frac{v_0}{\omega_n}\sin\omega_n t + x_0\cos\omega_n t + \frac{F_0}{2m\omega_n} t\sin\omega_n t$$