Harmonic Excitation of Damped Systems

Section 2.2

Motorcycle Engine Vibration Problem

- A motorcycle engine turns (and vibrates) at **300 rpm** with a harmonic force of **20 N**.

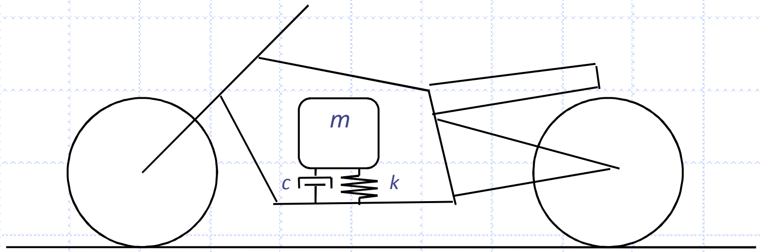
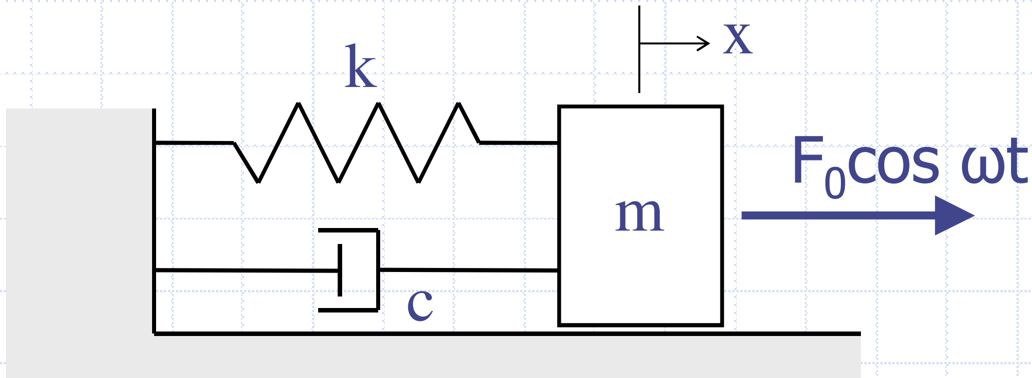


Figure 1. Motor vibrating in motorcycle

- What is the amplitude of the vibration with respect to the frame (assumed to be stationary) and phase of the response (with respect to the force), if:
  - the mass of the engine is **40 kg**,
  - the stiffness of the mounts is **40 kN/m** and
  - the damping coefficient is **125 Ns/m**?

Damped Spring-Mass System with Forced Vibration



$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t$$

Dividing both sides by *m* yields:

$$\ddot{x} + 2\varsigma\omega_n \dot{x} + \omega_n^2 x = f_0 \cos \omega t$$

where: $$f_0 = \frac{F_0}{m}$$

Solving the Differential Equation

Assume a particular solution:

$$x_p(t) = A\cos\omega t + B\sin\omega t$$

so that:

$$\dot{x}_p(t) = -\omega A\sin\omega t + \omega B\cos\omega t$$

$$\ddot{x}_p(t) = -\omega^2\left(A\cos\omega t + B\sin\omega t\right)$$

Substituting into the D.E.:

$$-\omega^2\left(A\cos\omega t + B\sin\omega t\right) + 2\varsigma\omega_n\omega\left(-A\sin\omega t + B\cos\omega t\right)$$
$$+\omega_n^2\left(A\cos\omega t + B\sin\omega t\right) = f_0\cos\omega t$$

Solving the Differential Equation

Separating cos ω*t* and sin ω*t* terms yields:

$$\left(-\omega^2 A + 2\varsigma\omega_n\omega B + \omega_n^2 A - f_0\right)\cos\omega t$$
$$+\left(-\omega^2 B - 2\varsigma\omega_n\omega A + \omega_n^2 B\right)\sin\omega t = 0$$

In order to equal zero all the time, both of the constants in front of the cos ω*t* and sin ω*t* terms must equal zero. I.e.:

$$\left(\omega_n^2 - \omega^2\right)A + 2\varsigma\omega_n\omega B = f_0$$
$$\left(-2\varsigma\omega_n\omega\right)A + \left(\omega_n^2 - \omega^2\right)B = 0$$

Solving the Differential Equation

- Solving for *A* and *B* yields:

$$A = \frac{\left(\omega_n^2 - \omega^2\right) f_0}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\varsigma\omega_n\omega\right)^2} \qquad B = \frac{2\varsigma\omega_n\omega f_0}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\varsigma\omega_n\omega\right)^2}$$

- Substituting for *A* and *B* in $x_p(t)$ yields:

$$x_p(t) = \frac{f_0}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\varsigma\omega_n\omega\right)^2}\left(\left(\omega_n^2 - \omega^2\right)\cos\omega t + 2\varsigma\omega_n\omega\sin\omega t\right)$$

Solving Differential Equation

- Re-writing in terms of X and θ instead of A and B yields:

$$x_p(t) = \frac{f_0}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\varsigma\omega_n\omega\right)^2}} \cos\left(\omega t - \tan^{-1}\left(\frac{2\varsigma\omega_n\omega}{\omega_n^2 - \omega^2}\right)\right)$$

- Final equation (for **forced vibration** without free vibration):

$$x_p(t) = X\cos\left(\omega t - \theta\right)$$

where:

$$X = \frac{f_0}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\varsigma\omega_n\omega\right)^2}}$$

$$\theta = \tan^{-1}\left(\frac{2\varsigma\omega_n\omega}{\omega_n^2 - \omega^2}\right)$$

Solving Differential Equation

- If we consider both the **free vibration** and **forced vibration** $x(t) = x_h(t) + x_p(t)$, solving for amplitude components in terms of x_0 and v_0 yields:

$$x(t) = e^{-\varsigma\omega_n t}\left\{\begin{aligned}&\left(x_0 - \frac{f_0\left(\omega_n^2-\omega^2\right)}{\left(\omega_n^2-\omega^2\right)^2+\left(2\varsigma\omega_n\omega\right)^2}\right)\cos\omega_d t\\&+\left(\frac{\varsigma\omega_n}{\omega_d}\left(x_0 - \frac{f_0\left(\omega_n^2-\omega^2\right)}{\left(\omega_n^2-\omega^2\right)^2+\left(2\varsigma\omega_n\omega\right)^2}\right) - \frac{2\varsigma\omega_n\omega^2 f_0}{\omega_d\left[\left(\omega_n^2-\omega^2\right)^2+\left(2\varsigma\omega_n\omega\right)^2\right]}+\frac{v_0}{\omega_d}\right)\sin\omega_d t\end{aligned}\right\}$$

$$+\frac{f_0}{\left(\omega_n^2-\omega^2\right)^2+\left(2\varsigma\omega_n\omega\right)^2}\left[\left(\omega_n^2-\omega^2\right)\cos\omega t + 2\varsigma\omega_n\omega\sin\omega t\right]$$

Plotting the "Frequency Response"

- Looking at the forced vibration $x_p(t)$, we can plot the amplitude X and phase lag θ as a function of forcing frequency ω.