Institute of Technology
West Virginia University

Numerical Simulation and Design

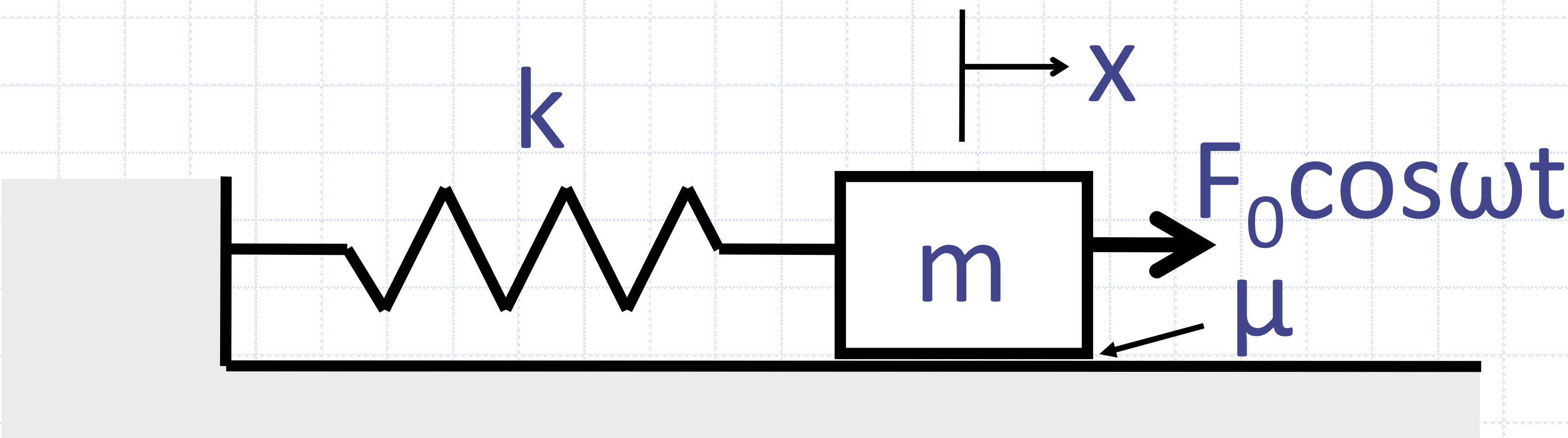
Section 2.8

Non-linear Response Properties

Section 2.9

Forced Vibration with Coulomb Damping

- A single degree-of-freedom system with mass 10 kg, spring stiffness of 1000 N/m and a Coulomb damping coefficient of 0.3 is excited by a harmonic force of 100 N amplitude at 100 rad/s.
- Plot the EXACT response x(t). Assume x_0 = -0.8683 mm and v_0 = 35 mm/s.



Section 2.8
Numerical Simulation & Design

- Recall that the "state-space" formulation for free vibration $m\ddot{x} + c\dot{x} + kx = 0$

is:

$$z_1(t) = x(t) \qquad z_2(t) = \dot{x}(t)$$

$$\xrightarrow[\text{wrt time}]{\text{taking derivative}}$$

$$\dot{z}_1 = \dot{x} = z_2$$

$$\dot{z}_2 = \ddot{x} = -\frac{k}{m}z_1 - \frac{c}{m}z_2$$

Putting into matrix form $\rightarrow$

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z}$$

Approximating

$$\dot{\mathbf{z}}(t_i) \approx \frac{\mathbf{z}(t_{i+1}) - \mathbf{z}(t_i)}{\Delta t}$$

$\rightarrow$

$$\mathbf{z}(t_{i+1}) = \mathbf{z}(t_i) + \Delta t \mathbf{A}\mathbf{z}(t_i)$$

Numerical Simulation & Design

- If we include the harmonic force term

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t$$

it becomes:

$$z_1(t) = x(t)$$
$$z_2(t) = \dot{x}(t)$$

$\xrightarrow[\text{wrt time}]{\text{taking derivative}}$

$$\dot{z}_1 = \dot{x} = z_2$$
$$\dot{z}_2 = \ddot{x} = -\frac{k}{m} z_1 - \frac{c}{m} z_2 + \frac{F_0}{m} \cos \omega t$$

Putting into matrix form $\rightarrow$

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{F_0}{m} \cos \omega t \end{bmatrix}$$

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{f}(t)$$

Approximating

$$\dot{\mathbf{z}}(t_i) \approx \frac{\mathbf{z}(t_{i+1}) - \mathbf{z}(t_i)}{\Delta t}$$

$\rightarrow$

$$\mathbf{z}(t_{i+1}) = \mathbf{z}(t_i) + \Delta t \mathbf{A}\mathbf{z}(t_i) + \Delta t \mathbf{f}(t_i)$$

Section 2.9
Non-linear Response Properties

- General formulation for any spring and damper forces:

$$\ddot{x}(t) + f[x(t), \dot{x}(t)] = f_0 \cos \omega t$$

- Setting up "state space" numerical time-integration:

$$\begin{aligned} z_1(t) &= x(t) \\ z_2(t) &= \dot{x}(t) \end{aligned} \xrightarrow[\text{wrt time}]{\text{taking derivative}} \begin{aligned} \dot{z}_1 &= \dot{x} = z_2 \\ \dot{z}_2 &= \ddot{x} = -f[z_1, z_2] + f_0 \cos \omega t \end{aligned}$$

Putting into matrix form →

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} z_2(t) \\ -f(z_1, z_1) \end{bmatrix} + \begin{bmatrix} 0 \\ f_0 \cos \omega t \end{bmatrix}$$

$$\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z}) + \mathbf{f}(t)$$

Approximating

$$\dot{\mathbf{z}}(t_i) \approx \frac{\mathbf{z}(t_{i+1}) - \mathbf{z}(t_i)}{\Delta t} \rightarrow \mathbf{z}(t_{i+1}) = \mathbf{z}(t_i) + \mathbf{F}[\mathbf{z}(t_i)]\Delta t + \mathbf{f}(t_i)\Delta t$$

Non-linear Response Properties

- For Coulomb damping:

$$m\ddot{x}(t) + \mu N \operatorname{sgn}(\dot{x}) + kx = F_0 \cos \omega t$$

- Setting up "state space" numerical time-integration solution:

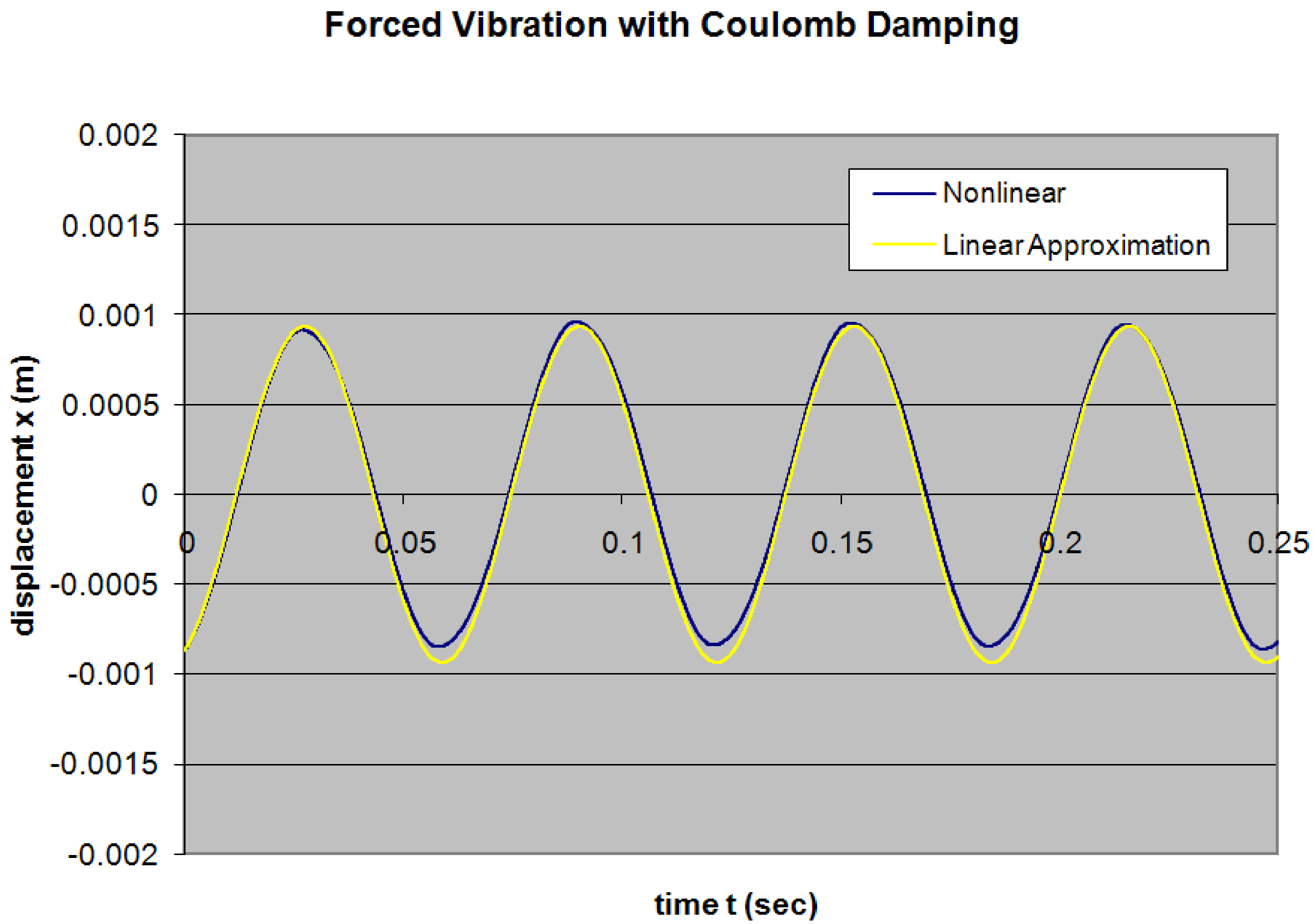
$$\begin{aligned} z_1(t) &= x(t) \\ z_2(t) &= \dot{x}(t) \end{aligned} \quad \xrightarrow[\text{wrt time}]{\text{taking derivative}} \quad \begin{aligned} \dot{z}_1 &= \dot{x} = z_2 \\ \dot{z}_2 &= \ddot{x} = -\frac{k}{m} z_1 - \frac{\mu N}{m} \operatorname{sgn}(z_2) + \frac{F_0}{m} \cos \omega t \end{aligned}$$

$$\begin{bmatrix} z_1(t_{i+1}) \\ z_2(t_{i+1}) \end{bmatrix} = \begin{bmatrix} z_1(t_i) \\ z_2(t_i) \end{bmatrix} + \begin{bmatrix} z_2(t_i) \\ -\frac{k}{m} z_1(t_i) - \frac{\mu N}{m} \operatorname{sgn}(z_2(t_i)) \end{bmatrix} \Delta t + \begin{bmatrix} 0 \\ \frac{F_0}{m} \cos(\omega t_i) \end{bmatrix} \Delta t$$

Numerical (Actual) Nonlinear Response compared with Linearized Response

- Actual problem using $F_{damping} = \mu mg \operatorname{sgn}(\dot{x})$ solved numerically
- Linearized problem using $c_{eq} = \dfrac{4\mu mg}{\pi\omega X}$ solved algebraically.

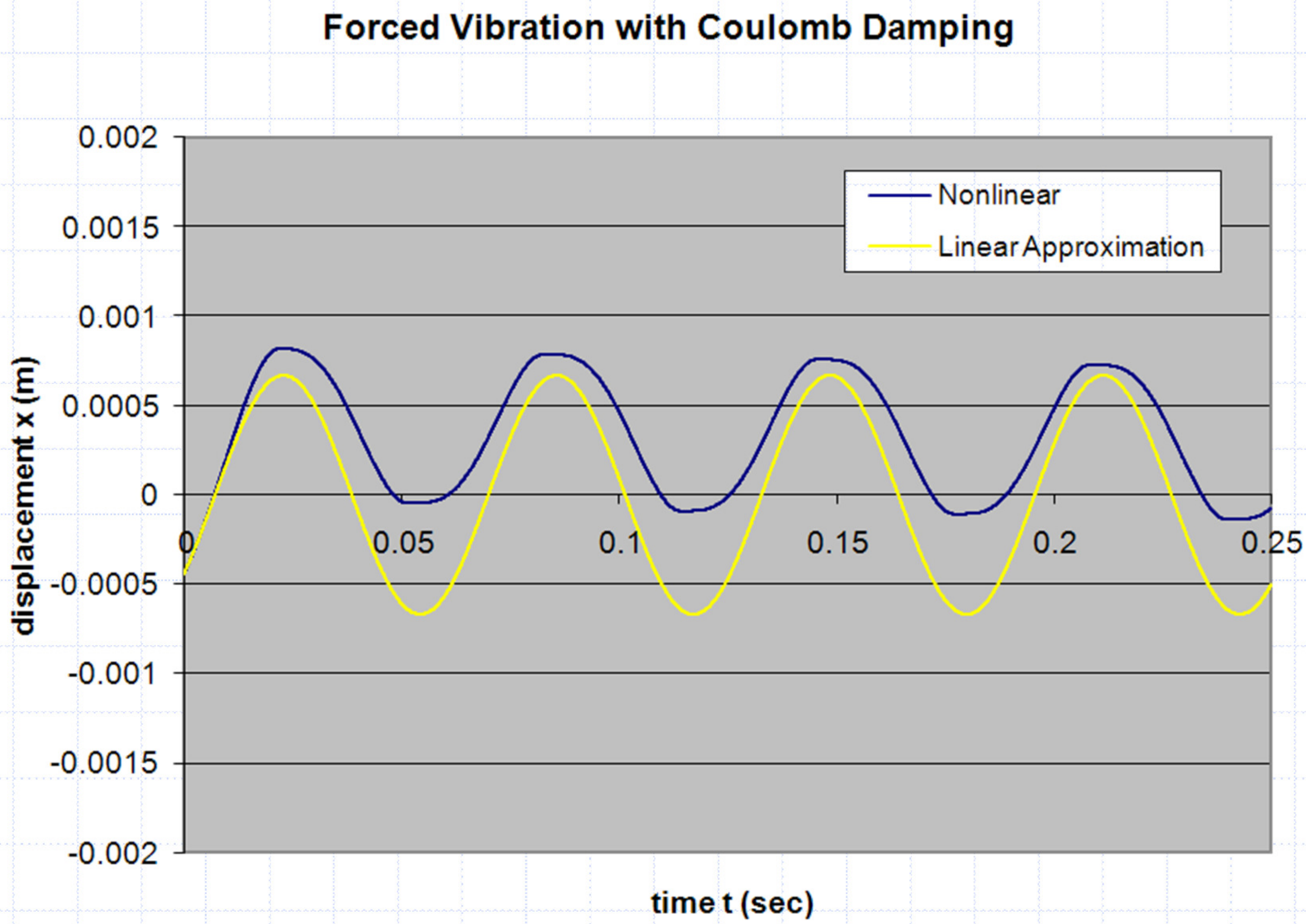
k	1000	N/m
m	10	kg
μ	0.3	
F0	100	N
g	9.81	m/s^2
ω	100	rad/s



Numerical (Actual) Nonlinear Response compared with Linearized Response

- Same as above with higher friction coefficient.

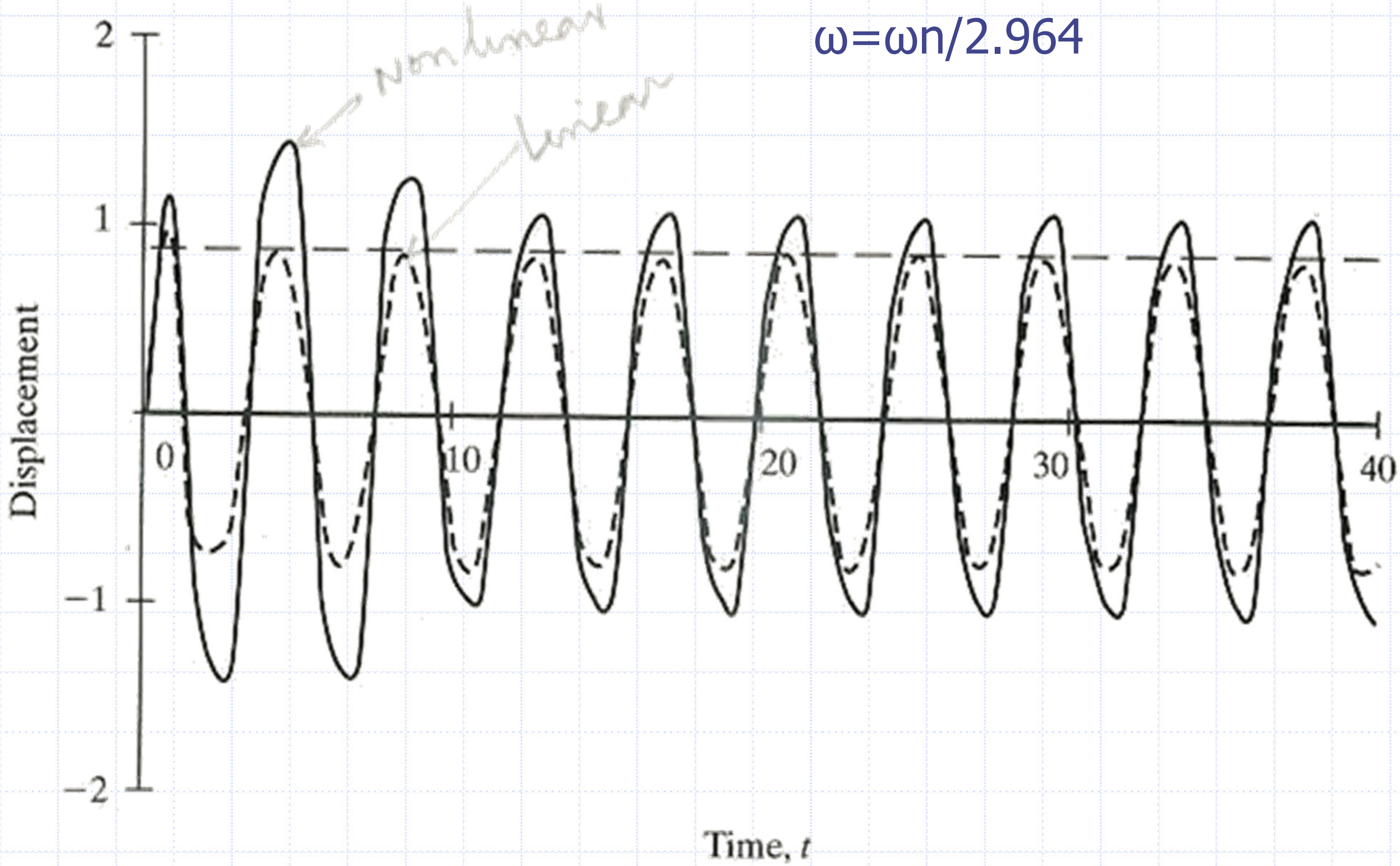
k	1000	N/m
m	10	kg
μ	0.6	
F0	100	N
g	9.81	m/s^2
ω	100	rad/s



Comparing Actual Nonlinear Response with Linearized Response

- Non-linear spring ($F_{spring} = kx - k_1 x^3$) versus linear spring ($F_{spring} = kx$)

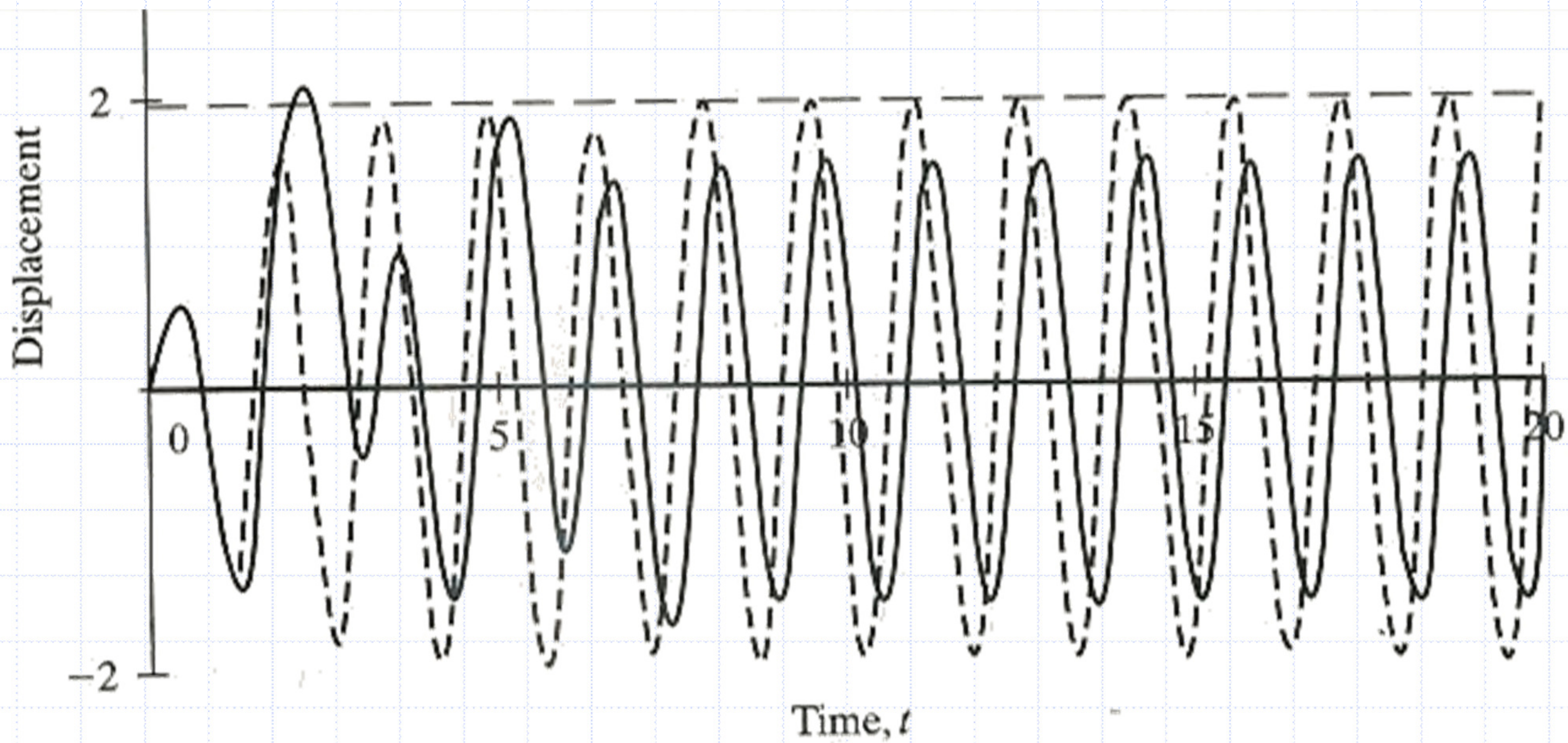
$\omega = \omega n/2.964$



Comparing Actual Nonlinear Response with Linearized Response

- Non-linear spring ($F_{spring} = kx - k_1 x^3$) versus linear spring ($F_{spring} = kx$)

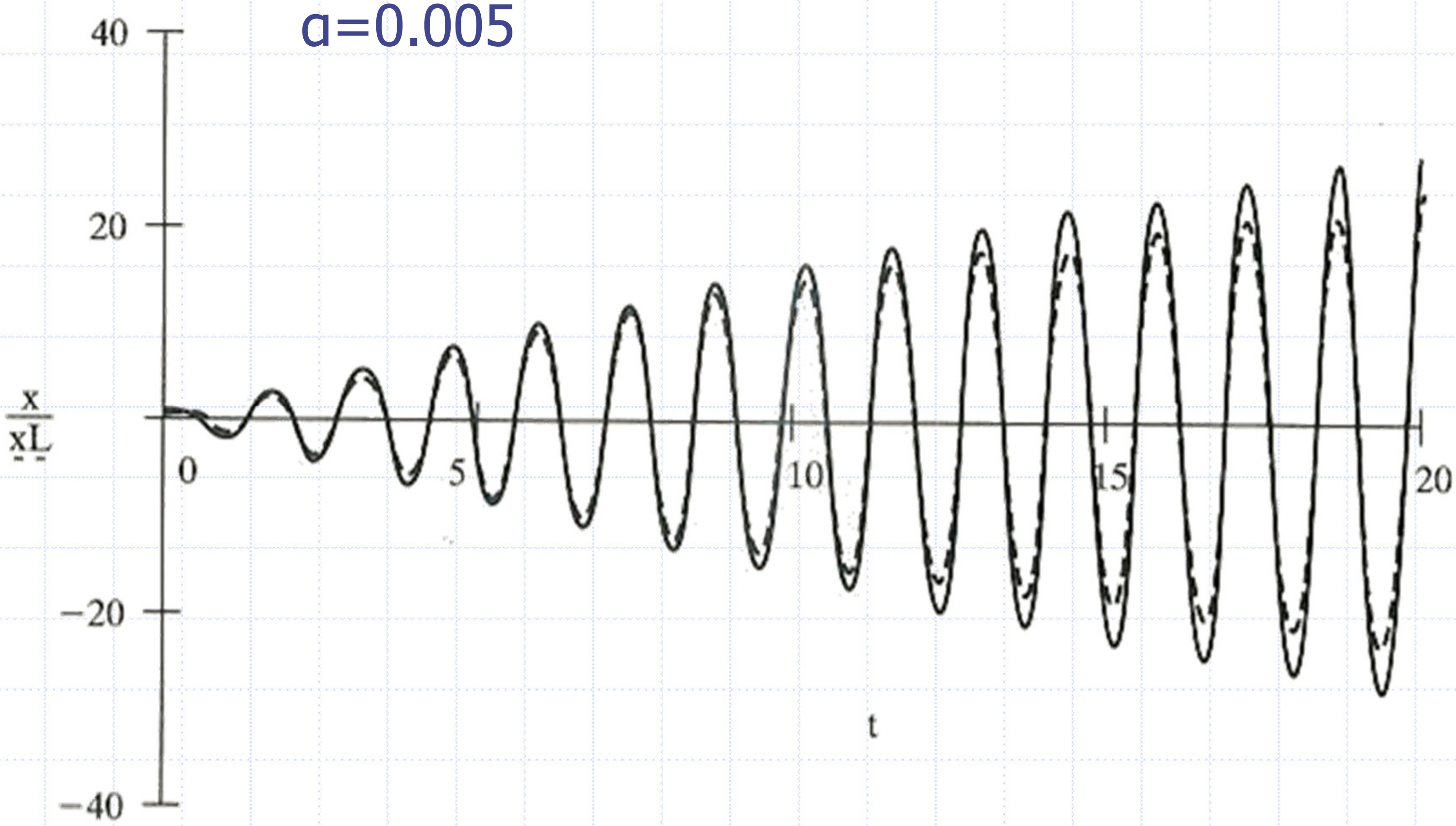
ω=ωn/1.09



Comparing Actual Nonlinear Response with Linearized Response

- Displacement-squared damping ($F_{damping} = \alpha \operatorname{sgn}(\dot{x})\dot{x}^2$) versus use of c_{eq} ($c_{eq} = \dfrac{4dX}{3\pi\omega}$)

α=0.005



Comparing Actual Nonlinear Response with Linearized Response

- Displacement-squared damping ($F_{damping} = \alpha\,\mathrm{sgn}(\dot{x})\dot{x}^2$) versus use of c_{eq} ($c_{eq} = \dfrac{4dX}{3\pi\omega}$)

α=0.5