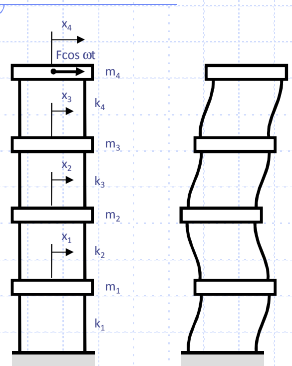
# Multi-Degree-of-Freedom Systems

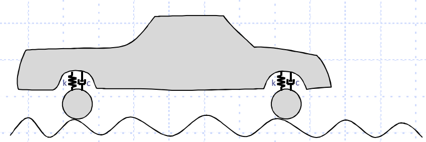
## Section 4.1



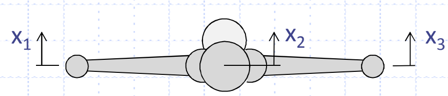
# Examples of Multi-Degree-of-Freedom (MDOF) Systems




Swaying Building



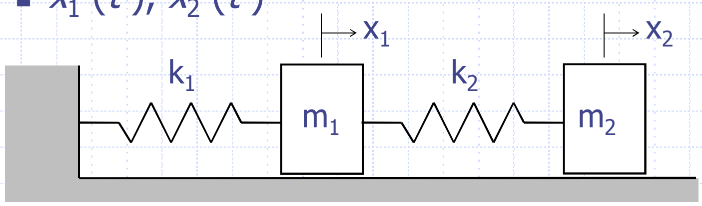
Moving Vehicle




Flying Aircraft

# Two DOF Spring-Mass System

- Given:
  - $m_1 = 9$ kg
  - $m_2 = 1$ kg
  - $k_1 = 24$ N/m
  - $k_2 = 3$ N/m
  - $x_{1,0} = 1$ mm
  - $x_{2,0} = 0$ mm
  - $v_{1,0} = 0$ m/s
  - $v_{2,0} = 0$ m/s
- Find:
  - $\omega_{n1}$, $\omega_{n2}$
  - $x_1(t)$, $x_2(t)$






# Two DOF Spring-Mass System

- Sol'n:

Step 1: Derive system differential equations

Mass 1

Mass 2

# Two DOF Spring-Mass System

## Write out coupled equations as matrix equation

With variables:

With numbers:



# Solving Differential Equation

## Step 2: Solve matrix differential equation

Assume a solution: $\bar{x}(t) = \bar{u}\sin(\omega_n t + \phi)$

$$\bar{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} u_1 \sin(\omega_n t + \phi) \\ u_2 \sin(\omega_n t + \phi) \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \sin(\omega_n t + \phi)$$

$$\ddot{\bar{x}}(t) = \begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix} =$$

# Solving Differential Equation

Substitute back into system differential equation:

Simplify:

$\overline{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a trivial solution; we are not interested in it.



# Solving Differential Equation

If we could solve for $\left(-\omega_n^2 \overline{M} + \overline{K}\right)^{-1}$ then we could get $\overline{u} = \left(-\omega_n^2 \overline{M} + \overline{K}\right)^{-1} \overline{0} = \overline{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

but this is still the trivial solution!

The only way to get a non-trivial solution would be if it were impossible to solve for $\left(-\omega_n^2 \overline{M} + \overline{K}\right)^{-1}$

When is a matrix not invertible??

# Solving Differential Equation

## When the determinant is zero!!

Need to find $\omega_n$ that satisfy: $\det\left(-\omega_n^2\overline{M}+\overline{K}\right)=0$

Recall that the determinant of a matrix is computed as follows:



# Solving Differential Equation

Putting in the numbers:

# Mode 1: $\omega_{n1}$ =

Substituting $\omega_{n1}$ into equation from slide 7:



# Mode 1

These equations are **not linearly independent**. There is no single solution. Instead there is a set of solutions. But each solution must satisfy:

We could use a scaling factor (*s*):

This is the **"Mode Shape"**.

# Mode 1

Usually *s* is chosen so that the largest u value is 1:

This is the **"Normalized Mode Shape"**.

Therefore, the equation of motion, considering only the first mode is:



# Mode 2: $\omega_{n2}$ =

Substituting $\omega_{n2}$ into equation from slide 7:

# Mode 2

The normalized mode shape is:

The equation of motion, considering only the second mode is:



# Both Modes Together

The overall equation of motion, considering both modes is:

The unknowns ($A_1$, $\Phi_1$, $A_2$, $\Phi_2$) must be determined from the initial conditions.

# Solving for $A_1$, $\Phi_1$, $A_2$, $\Phi_2$

- Applying the initial conditions:

$$\bar{x}(t) =$$

$$\dot{\bar{x}}(t) =$$

$$\bar{x}(0) =$$

$$\dot{\bar{x}}(0) =$$



# Solving for $A_1$, $\Phi_1$, $A_2$, $\Phi_2$

- Need to solve the nonlinear equations:

# Solving for $A_1$, $\Phi_1$, $A_2$, $\Phi_2$



# Solving for $A_1$, $\Phi_1$, $A_2$, $\Phi_2$

- Therefore, the overall equation of motion is: