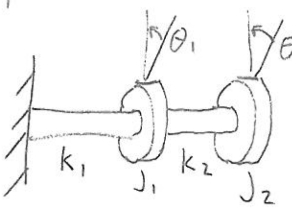
Problem 4.~~11~~ 12

Given:



$k_1 = k_2 = k$

$J_1 = 3J_2 = 3J$

Find: i) System differential equation in matrix form

ii) Natural frequencies

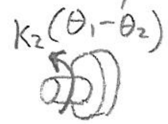
iii) Mode shapes

Sol'n: i) Free body diagram for J_1 body

$$\sum M = -k_1\theta_1 + k_2(\theta_2-\theta_1) = J_1\ddot{\theta}_1$$

$$J_1\ddot{\theta}_1 + (k_1+k_2)\theta_1 - k_2\theta_2 = 0$$

Free body diagram for J_2 body



$$\sum M = k_2(\theta_1-\theta_2) = J_2\ddot{\theta}_2$$

$$J_2\ddot{\theta}_2 - k_2\theta_1 + k_2\theta_2 = 0$$

In matrix form:

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Putting in terms of J & k

$$\begin{bmatrix} 3J & 0 \\ 0 & J \end{bmatrix}\ddot{\bar{\theta}} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\bar{\theta} = \bar{0}$$

ii) To find natural frequencies

$$\det\left(-\omega_n^2\bar{M} + \bar{K}\right) = 0$$

$$\det\left(-\omega_n^2\begin{bmatrix} 3J & 0 \\ 0 & J \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\right) = \det\left(\begin{bmatrix} 2k-3J\omega_n^2 & -k \\ -k & k-J\omega_n^2 \end{bmatrix}\right) = 0$$

$$(2k-3J\omega_n^2)(k-J\omega_n^2) - (-k)(-k) = 0$$

$$3J^2\omega_n^4 - 5Jk\omega_n^2 + k^2 = 0$$

divide by J^2

$$3\omega_n^4 - 5\frac{k}{J}\omega_n^2 + \left(\frac{k}{J}\right)^2 = 0$$

$$\omega_n^2 = \frac{+5\frac{k}{J} \pm \sqrt{\left(-5\frac{k}{J}\right)^2 - 4(3)\left(\frac{k}{J}\right)^2}}{2(3)} = \frac{5 \pm \sqrt{(-5)^2 - 4(3)}}{6}\,\frac{k}{J}$$

$$\omega_{n_1} = \sqrt{0.232\left(\frac{k}{J}\right)} = \boxed{0.482\sqrt{\frac{k}{J}}}$$

$$\omega_{n_2} = \sqrt{1.434\left(\frac{k}{J}\right)} = \boxed{1.198\sqrt{\frac{k}{J}}}$$

iii) Mode 1: $\omega_{n_1} = 0.482\sqrt{\frac{k}{J}}$ rad/s

$$\left(-\omega_{n_1}^2 \bar{M} + \bar{K}\right)\bar{u}_1 = \bar{0}$$

$$\left(-0.482^2\frac{k}{J}\begin{bmatrix}3J & 0\\ 0 & J\end{bmatrix} + \begin{bmatrix}2k & -k\\ -k & k\end{bmatrix}\right)\bar{u}_1 = \bar{0}$$

$$\begin{bmatrix}2k - 3(0.482)^2k & -k\\ -k & k-(0.482)^2k\end{bmatrix}\bar{u}_1 = \bar{0}$$

Divide both side by k

$$\begin{bmatrix}1.303 & -1\\ -1 & 0.767\end{bmatrix}\begin{Bmatrix}u_{1,1}\\ u_{2,1}\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix} \Rightarrow \text{using row 2}$$

$$-u_{1,1} + 0.767\,u_{2,1} = 0$$

$$\boxed{\bar{u}_1 = \begin{Bmatrix}0.767\\ 1\end{Bmatrix}}$$

Mode 2: $\omega_{n_2} = 1.198\sqrt{\frac{k}{J}}$ rad/s

$$\left(-1.198^2\frac{k}{J}\begin{bmatrix}3J & 0\\ 0 & J\end{bmatrix} - \begin{bmatrix}2k & -k\\ -k & k\end{bmatrix}\right)\bar{u}_2 = \bar{0}$$

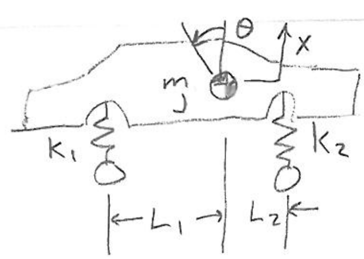
$$\begin{bmatrix}-2.303 & -1\\ -1 & -0.434\end{bmatrix}\begin{Bmatrix}u_{1,2}\\ u_{2,2}\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix} \Rightarrow \text{from row 2}$$

$$-u_{1,2} - 0.434\,u_{2,2} = 0$$

$$\boxed{\bar{u}_2 = \begin{Bmatrix}-0.434\\ 1\end{Bmatrix}}$$

Homework 4.~~12~~ 13 +

Given:



$k_1 = k_2 = 20\,kN/m$
$L_1 = 1.4\,m$
$L_2 = 0.9\,m$
$m = 4000\,kg$
$J = 2560\,kg\cdot m^2$

Find: i) System diff. eq.
ii) Nat. freq.
iii) Mode shapes.